Turbulence in Compressible Flows

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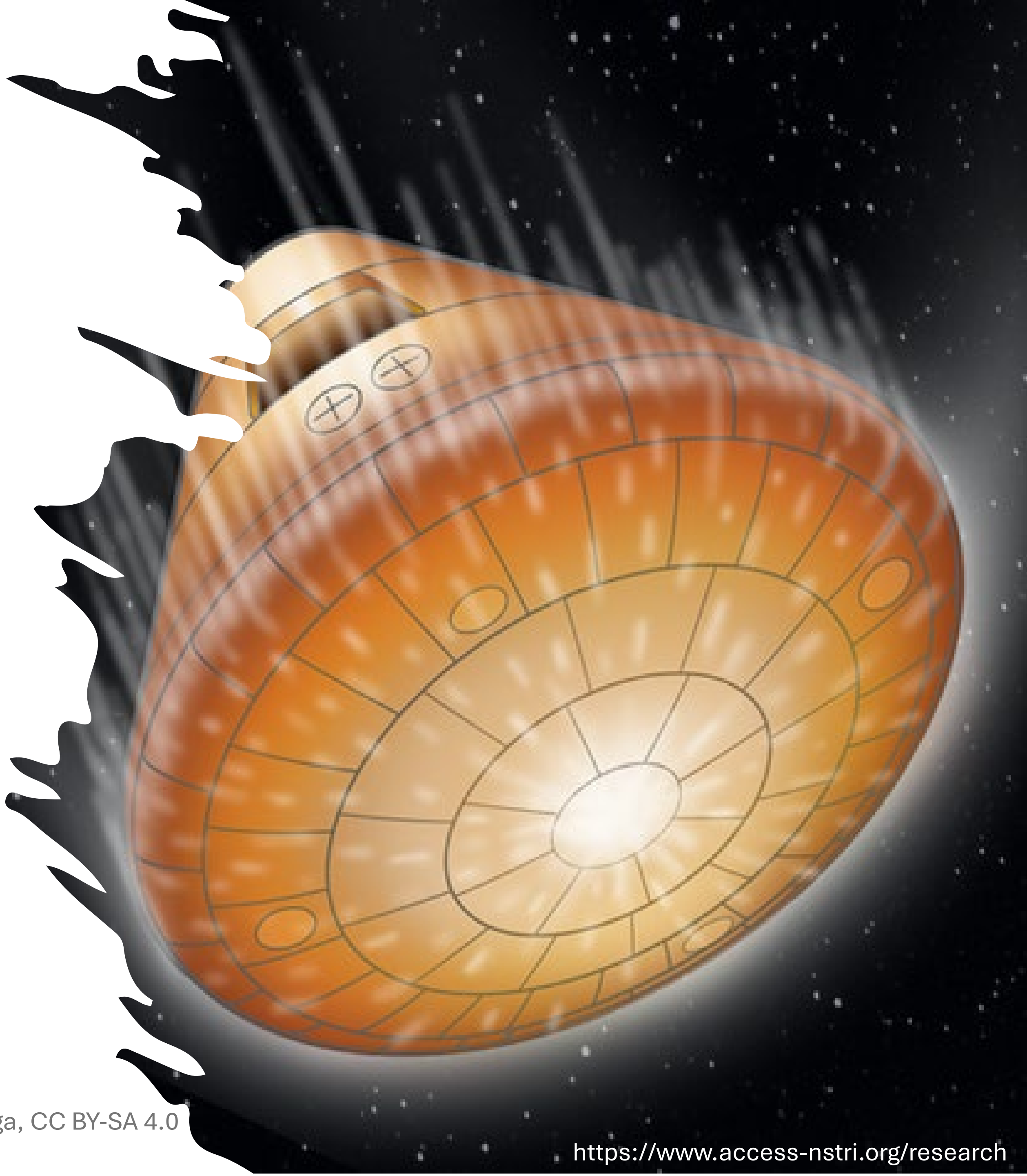
Outline

- Compressibility Review
- Motivation
- Favre Averaging
- Favre-Averaged Gov. Eqns.
- Compressible Flow Closures
- Shock - Boundary Layer Interaction
- Relationship to Incompressible Turbulence
- Energy/Temperature Considerations
- Supplemental References


https://www.access-nstri.org/research

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Source: Modern Compressible Flow With Historical Perspective 3rd Ed., John D. Anderson Jr., McGraw-Hill, 2003

Compressibility Review

- All fluids are compressible, some more than others

- Density varies – there is always a fractional change in volume (density) of a fluid element per unit change in pressure:

$$\tau = \frac{-1}{v}\frac{dv}{dp} = \frac{1}{\rho}\frac{d\rho}{dp} \rightarrow d\rho = \rho\tau dp$$

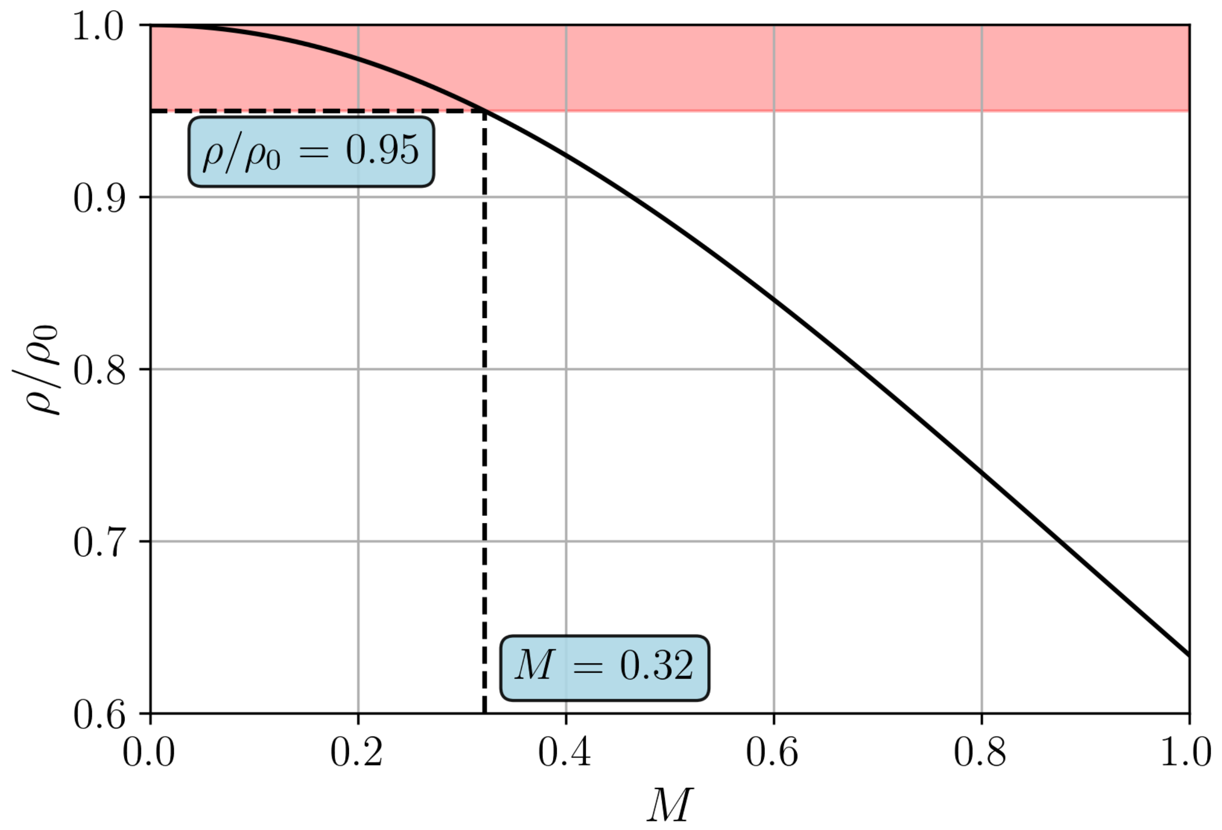
- Compressibility is a property of the fluid
  - Liquids like water: $\tau = \mathcal{O}(10^{-10})$ m²/N
  - Gases like air: $\tau = \mathcal{O}(10^{-5})$ m²/N

- Moving fluid leads to pressure gradients dp
  - For liquids, because τ is small, the resulting change in density is negligible
  - For gases, the pressure gradients can lead to substantial changes in the density ($\geq$ 5%)

Source: NASA Isentropic Flow Calorically Perfect Gas
https://www.grc.nasa.gov/www/BGH/isentrop.html

Compressibility Review

- Liquids are assumed incompressible (constant density)
- Gases are considered compressible if Mach number greater than 0.3
- Example: Air accelerated from rest isentropically, $\frac{\rho}{\rho_0} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{-1/(\gamma-1)}$



The focus for this lecture are gasses with $M > 0.3$

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Compressibility Review - Turbulent Mach Number

- Since density is not constant, you will have a mean variation in density in addition to density fluctuations: $\rho = \overline{\rho} + \rho'$ or $\rho = \tilde{\rho} + \rho''$
- Define a turbulence Mach number M_t based on the turbulence fluctuations:

$$M_t(x_i) = \frac{u'_{i,RMS}(x_i)}{\overline{a}(x_i)} = \frac{\sqrt{\overline{u_i'^2}}}{\sqrt{\gamma R_{gas}\overline{T}}} = \frac{\sqrt{\overline{u_i' u_i'}}}{\sqrt{\gamma R_{gas}\overline{T}}} \quad \text{or} \quad M_t(x_i) = \frac{u''_{i,RMS}(x_i)}{\widetilde{a}(x_i)} = \frac{\sqrt{\widetilde{u_i''^2}}}{\sqrt{\gamma R_{gas}\widetilde{T}}} = \frac{\sqrt{\overline{\rho u_i'' u_i''}/\overline{\rho}}}{\sqrt{\gamma R_{gas}\widetilde{T}}}$$

- Compressibility effects beyond mean-property variations are governed primarily by the turbulent Mach number
- Compressible turbulence is when $M_t \gtrsim 0.3$. Dilatational and thermodynamic fluctuations (ρ', T', μ') become significant (should account for this in models)
- These considerations may become important in hypersonic flows, shock-boundary layer interactions, and other highly compressible flows

Compressibility Review - Turbulent Mach Number

- Since density is not constant, you will have a mean variation in density in addition to density fluctuations: $\rho = \overline{\rho} + \rho'$ or $\rho = \tilde{\rho} + \rho''$

mean compressibility $\neq$ turbulent compressibility

aka

turbulence in compressible flow $\neq$ compressible turbulence

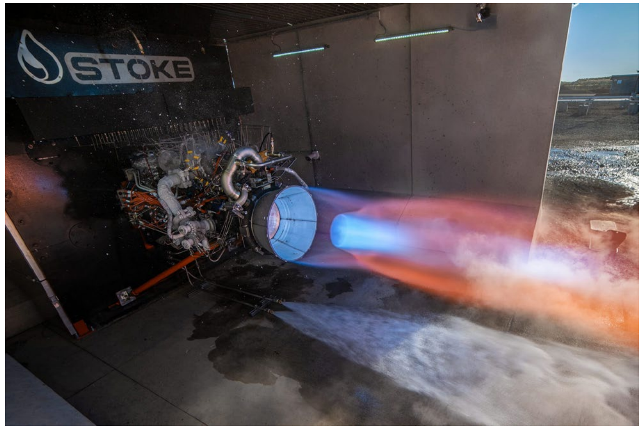
- These considerations may become important in hypersonic flows, shock-boundary layer interactions, and other highly compressible flows

Motivation – Why study turbulence in compressible flows?

- High speed, compressible, turbulent flows are ubiquitous!
  - Industrial systems
  - Propulsion systems
  - Turbomachinery
  - Aviation (civil and defense)
  - Rocketry
  - Nozzles and jets
  - Hypersonics

Boeing 777
Source: https://www.boeing.com/commercial/777

Artist rendering of KREPE-2 capsule re-entry
Source: https://www.nasa.gov/learning-resources/student-built-capsules/

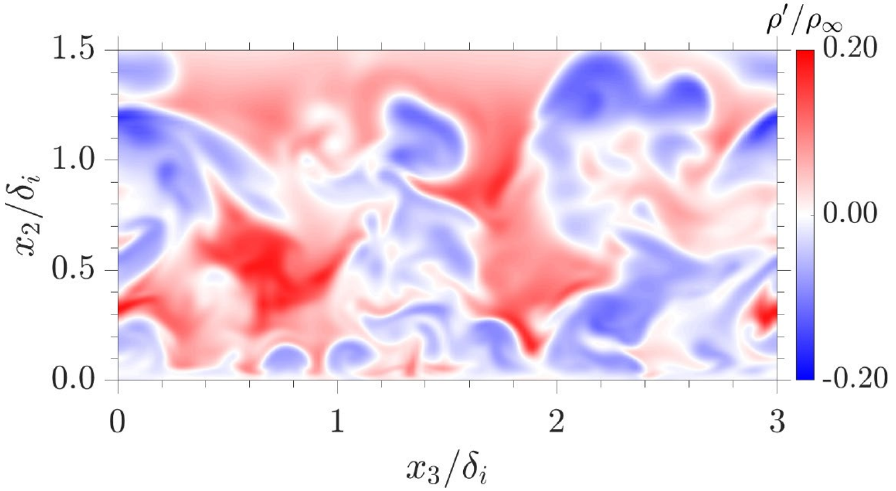


Stoke Space rocket engine
Source: https://www.stokespace.com/stoke-space-completes-first-successful-hotfire-test-of-full-flow-staged-combustion-engine/

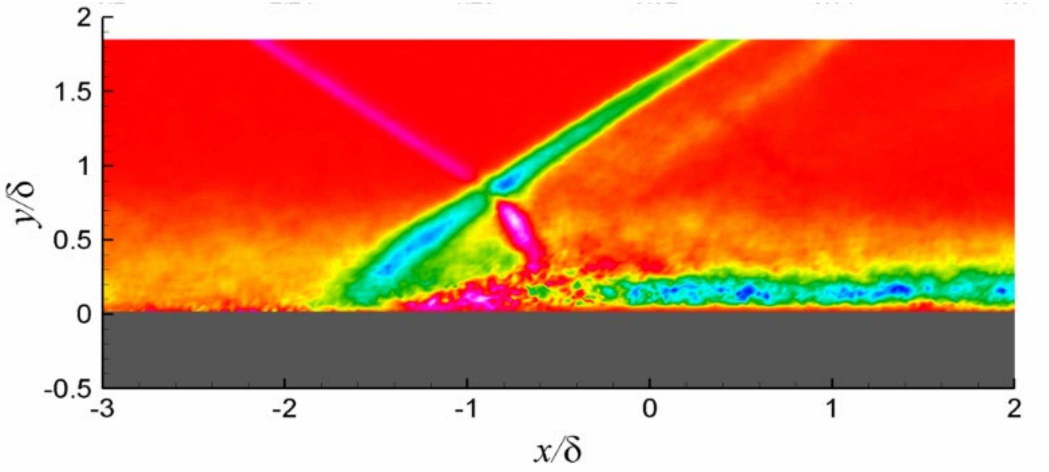
Central steam plant
Source: https://thermon.com/precision-boilers/markets/central-steam-plants/

Motivation – Why study turbulence in compressible flows?

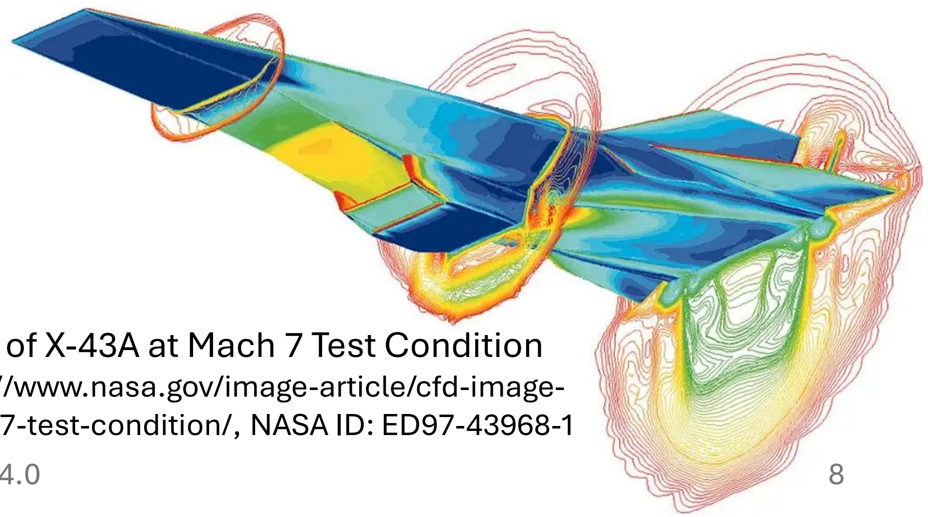
- Occurs in both the subsonic and supersonic flow regimes
- Active research areas (not exhaustive):
  - Velocity transformations
  - Temperature transformations
  - Velocity-temperature relationships
  - Turbulence modeling and closures
  - Shock-boundary layer interactions (SBLI)



Density fluctuations up to 20% of freestream density value for Mach 2.5 adiabatic wall boundary layer



PIV measurement of Reynolds shear stress distribution in SBLI
Source: Humble et al. 13th Int. Symp. on Applications of Laser Techniques to Fluid Mechanics Lisbon, Portugal, 26 – 29 June, 2006. Paper No. 1033



CFD Image of X-43A at Mach 7 Test Condition
Source: https://www.nasa.gov/image-article/cfd-image-of-x-43a-mach-7-test-condition/, NASA ID: ED97-43968-1

Density Weighted (Favre) Averaging

- Compressible flows are characterized by density variation
  - Must now account for density and temperature fluctuations in addition to velocity and pressure fluctuations
- Density weighted (Favre) averaging simplifies the governing equations
  - Simple Reynolds averaging results in extra density fluctuation correlation terms
- The Favre averaged equations recover a form reminiscent of the well-known incompressible equations
  - Can “reuse” incompressible turbulence theory (turb. modeling, law of the wall, etc.)
- Notation:

Reynolds average $\overline{(\cdot)}$ **Favre average** $\widetilde{(\cdot)}$

- In general, the Reynolds average is an ensemble average
  - If statistically stationary can be time average
  - If homogeneous can be spatial average

Density Weighted (Favre) Averaging

Common scenarios that use Favre averaging:

1. When density fluctuations are non-negligible
   - High Mach number flows
   - Strong heating / cooling
   - Combustion

2. Working with Reynolds Averaged Navier-Stokes (RANS) of compressible flows
   - Needs compressible closures

Density Weighted (Favre) Averaging

Reynolds decomposition: $\phi = \overline{\phi} + \phi'$, where $\overline{\phi'} = 0$

Favre decomposition: $\phi = \widetilde{\phi} + \phi''$

Favre average: $\widetilde{\phi} = \dfrac{\overline{\rho\phi}}{\overline{\rho}}$ or $\widetilde{\phi} = \overline{\phi} + \dfrac{\overline{\rho'\phi'}}{\overline{\rho}}$

Favre averaging is not a modeling assumption. The definition of the Favre average absorbs the density-velocity (pressure, temperature, etc.) correlations.

Density Weighted (Favre) Averaging

Reynolds decomposition: $\phi = \overline{\phi} + \phi'$, where $\overline{\phi'} = 0$

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For density and pressure we use the Reynolds Decomposition.

For all other primitive variables we use the Favre Decomposition.

Density Weighted (Favre) Averaging

Averaging Rules: $\overline{\phi + \psi} = \overline{\phi} + \overline{\psi}$

$$\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \overline{\phi}}{\partial x}$$

$$\overline{\overline{\phi}\psi} = \overline{\phi}\,\overline{\psi}$$

$$\overline{\widetilde{\phi}} = \widetilde{\phi}$$

$$\overline{\rho\widetilde{\phi}} = \overline{\rho}\widetilde{\phi}$$

$$\widetilde{\rho\phi} = \overline{\rho}\widetilde{\phi}$$

Density Weighted (Favre) Averaging

Reynolds average of the density weighted Favre fluctuations are zero

$$\overline{\rho\phi''} = 0$$

Reynolds average of the Favre fluctuation alone is not zero

$$\overline{\phi''} = \frac{-\overline{\rho'\phi'}}{\overline{\rho}} \neq 0$$

Triple product expansion

$$\overline{\rho\phi\psi} = \overline{\rho}\widetilde{\phi}\widetilde{\psi} + \overline{\rho\phi''\psi''}$$

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Favre-Averaged Governing Equations - Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$

$$\overline{\frac{\partial(\overline{\rho} + \rho')}{\partial t}} + \overline{\frac{\partial(\overline{\rho} + \rho')(\widetilde{u_j} + u_j'')}{\partial x_j}} = 0$$

$$\frac{\partial \overline{\overline{\rho}}}{\partial t} + \frac{\partial \overline{\rho'}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\overline{\rho}\widetilde{u_j}} + \overline{\overline{\rho} u_j''} + \overline{\rho' \widetilde{u_j}} + \overline{\rho' u_j''}\right) = 0$$

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho'}^{\,0}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\rho}\widetilde{u_j} + \overline{\rho'}^{\,0}\widetilde{u_j} + \overline{(\overline{\rho} + \rho')u_j''}\right) = 0$$

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\rho}\widetilde{u_j} + \overline{\rho u_j''}^{\,0}\right) = 0$$

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}\widetilde{u_j}}{\partial x_j} = 0$$

Similar in form to incompressible RANS continuity equation

Favre-Averaged Governing Equations - Continuity

$$\boxed{\frac{\partial \rho}{\partial t}+\frac{\partial \rho u_j}{\partial x_j}=0}$$

$$\frac{\overline{\partial(\overline{\rho}+\rho')}}{\partial t}+\frac{\overline{\partial(\overline{\rho}+\rho')(\overline{u_j}+u_j')}}{\partial x_j}=0$$

$$\frac{\partial\overline{\overline{\rho}}}{\partial t}+\frac{\partial\overline{\rho'}}{\partial t}+\frac{\partial}{\partial x_j}\left(\overline{\overline{\rho}\,\overline{u_j}}+\overline{\overline{\rho}u_j'}+\overline{\rho'\overline{u_j}}+\overline{\rho'u_j'}\right)=0$$

$$\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial\cancelto{0}{\overline{\rho'}}}{\partial t}+\frac{\partial}{\partial x_j}\left(\overline{\rho}\,\overline{u_j}+\overline{\rho}\cancelto{0}{\overline{u_j'}}+\cancelto{0}{\overline{\rho'}}\,\overline{u_j}+\overline{\rho'u_j'}\right)=0$$

$$\boxed{\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial\overline{\rho}\,\overline{u_j}}{\partial x_j}+\frac{\partial\overline{\rho'u_j'}}{\partial x_j}=0}$$

Extra density-velocity fluctuation correlation

Showing Reynolds decomposition to highlight benefit of Favre averaging which absorbed density fluctuation correlation

Favre-Averaged Governing Equations - Momentum

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial t_{ij}}{\partial x_j}$$

$$t_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

$$\lambda = -\frac{2}{3}\mu \qquad \text{(Stokes' hypothesis)}$$

Decompositions

$$\rho = \overline{\rho} + \rho'$$

$$P = \overline{P} + P'$$

$$\mu = \overline{\mu} + \mu'$$

$$\lambda = \overline{\lambda} + \lambda'$$

$$u_i = \widetilde{u}_i + u_i''$$

$$\frac{\partial(\overline{\rho}\widetilde{u}_i)}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{u}_i\widetilde{u}_j)}{\partial x_j} = -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial \overline{t_{ij}}}{\partial x_j} - \frac{\partial(\overline{\rho u_i'' u_j''})}{\partial x_j}$$

Favre-averaged Reynolds-stress tensor: $\tau_{ij} = \dfrac{-\overline{\rho u_i'' u_j''}}{\overline{\rho}}$

Favre-Averaged Governing Equations - Momentum

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$$u_i = \widetilde{u_i} + u_i''$$

Terms to model

$$\frac{\partial(\overline{\rho}\widetilde{u_i})}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{u_i}\widetilde{u_j})}{\partial x_j} = -\frac{\partial\overline{P}}{\partial x_i} + \frac{\partial\overline{t_{ij}}}{\partial x_j} - \frac{\partial(\overline{\rho u_i'' u_j''})}{\partial x_j}$$

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Favre-Averaged Governing Equations - Momentum

$$t_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

$$\lambda = -\frac{2}{3}\mu \qquad \text{(Stokes' hypothesis)}$$

$$\overline{t_{ij}} = \overline{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}}$$

$$\overline{t_{ij}} = \underbrace{\overline{\mu}\left(\frac{\partial \widetilde{u_i}}{\partial x_j} + \frac{\partial \widetilde{u_j}}{\partial x_i}\right) + \overline{\lambda}\frac{\partial \widetilde{u_k}}{\partial x_k}\delta_{ij}}_{\widetilde{t_{ij}}} + \underbrace{\overline{\mu\left(\frac{\partial u_i''}{\partial x_j} + \frac{\partial u_j''}{\partial x_i}\right)} + \overline{\lambda\frac{\partial u_k''}{\partial x_k}}\delta_{ij}}_{t_{ij}''}$$

$$\overline{t_{ij}} = \widetilde{t_{ij}} \qquad \left(\text{assuming } |\widetilde{t_{ij}}| \gg |t_{ij}''|\right)$$

<u>Decompositions</u>

$$\rho = \overline{\rho} + \rho'$$

$$P = \overline{P} + P'$$

$$\mu = \overline{\mu} + \mu'$$

$$\lambda = \overline{\lambda} + \lambda'$$

$$u_i = \widetilde{u_i} + u_i''$$

Favre-Averaged Governing Equations - Energy

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{u_i u_i}{2}\right)\right]+\frac{\partial}{\partial x_j}\left[\rho u_j\left(h+\frac{u_i u_i}{2}\right)\right]=\frac{\partial(u_i t_{ij})}{\partial x_j}-\frac{\partial q_j}{\partial x_j}$$

Specific internal energy and specific enthalpy

$$h = e + \frac{P}{\rho}$$

$$e = c_v T \quad \text{(calorically perfect, } c_v \text{ constant)}$$

$$h = c_p T \quad \text{(calorically perfect, } c_p \text{ constant)}$$

Heat Flux Vector from Fourier's Law and thermal conductivity

$$q_j = -\kappa \frac{\partial T}{\partial x_j} = -\frac{\mu}{Pr_L}\frac{\partial h}{\partial x_j}$$

$$q_{Lj} = -\kappa \overline{\frac{\partial T}{\partial x_2}}$$

Divergence of molecular heat flux shown here gives molecular transport (diffusion) of heat

$$\kappa = \frac{\mu c_p}{Pr_L} \quad (Pr_L \text{ is laminar Prandtl number})$$

Decompositions

$$\rho = \overline{\rho} + \rho'$$

$$P = \overline{P} + P'$$

$$\mu = \overline{\mu} + \mu'$$

$$\lambda = \overline{\lambda} + \lambda'$$

$$\kappa = \overline{\kappa} + \kappa'$$

$$u_i = \widetilde{u}_i + u_i''$$

$$e = \widetilde{e} + e''$$

$$h = \widetilde{h} + h''$$

$$T = \widetilde{T} + T''$$

$$q_j = q_{Lj} + q_j'$$

Favre-Averaged Governing Equations - Energy

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{u_i u_i}{2}\right)\right]+\frac{\partial}{\partial x_j}\left[\rho u_j\left(h+\frac{u_i u_i}{2}\right)\right]=\frac{\partial (u_i t_{ij})}{\partial x_j}-\frac{\partial q_j}{\partial x_j}$$

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e}+\frac{\tilde{u}_i\tilde{u}_i}{2}\right)+\frac{\overline{\rho u_i'' u_i''}}{2}\right]+\frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h}+\frac{\tilde{u}_i\tilde{u}_i}{2}\right)+\tilde{u}_j\frac{\overline{\rho u_i'' u_i''}}{2}\right]$$
$$=\frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\overline{t_{ij}}-\overline{\rho u_i'' u_j''}\right)\right]-\frac{\partial}{\partial x_j}\left[q_{Lj}+\overline{\rho u_j'' h''}-\overline{u_i'' t_{ij}}+\frac{\overline{\rho u_j'' u_i'' u_i''}}{2}\right]$$

Favre-averaged Reynolds-stress tensor: $\tau_{ij}=\dfrac{-\overline{\rho u_i'' u_j''}}{\bar{\rho}}$

Turbulent kinetic energy per unit volume: $k=\dfrac{\overline{\rho u_i'' u_i''}}{2\bar{\rho}}$

Turbulent heat flux: $q_{Tj}=\overline{\rho u_j'' h''}$

Decompositions

$\rho=\bar{\rho}+\rho'$

$P=\overline{P}+P'$

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$\kappa=\bar{\kappa}+\kappa'$

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$q_j=q_{Lj}+q_j'$

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$$=\frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\overline{t_{ij}}-\overline{\rho u_i'' u_j''}\right)\right]-\frac{\partial}{\partial x_j}\left[q_{Lj}+\overline{\rho u_j'' h''}-\overline{u_i'' t_{ij}}+\frac{\overline{\rho u_j'' u_i'' u_i''}}{2}\right]$$

Terms to model

Favre-averaged Reynolds-stress tensor: $\tau_{ij}=\frac{-\overline{\rho u_i'' u_j''}}{\bar{\rho}}$

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$u_i=\tilde{u}_i+u_i''$

$e=\tilde{e}+e''$

$h=\tilde{h}+h''$

$T=\tilde{T}+T''$

$q_j=q_{Lj}+q_j'$

Favre-Averaged Governing Equations - TKE

$$\rho u_i'' \frac{\partial u_i}{\partial t} + \rho u_i'' u_j \frac{\partial u_i}{\partial x_j} = -u_i'' \frac{\partial P}{\partial x_i} + u_i'' \frac{\partial t_{ij}}{\partial x_j}$$

$$\overline{\rho}\frac{\partial k}{\partial t} + \overline{\rho}\widetilde{u}_j \frac{\partial k}{\partial x_j} = \underbrace{\overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j}}_{P_k} \underbrace{- \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}}}_{\overline{\rho}\epsilon} + \underbrace{\frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j}}_{D} \underbrace{- \frac{\partial \overline{u_j'' P'}}{\partial x_j}}_{\Pi_t} \underbrace{- \frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j}}_{T_k} \underbrace{- \overline{u_i''}\frac{\partial \overline{P}}{\partial x_i}}_{\Pi_w} + \underbrace{\overline{P'\frac{\partial u_i''}{\partial x_i}}}_{\Pi_d}$$

<u>Decompositions</u>

$$\rho = \overline{\rho} + \rho'$$

$$P = \overline{P} + P'$$

$$\mu = \overline{\mu} + \mu'$$

$$\lambda = \overline{\lambda} + \lambda'$$

$$u_i = \widetilde{u}_i + u_i''$$

- P_k: Production
- $\overline{\rho}\epsilon$: Viscous dissipation
- D: Molecular (viscous) diffusion
- Π_t: Pressure diffusion
- T_k : Turbulent transport
- Π_w: Pressure work
- Π_d: Pressure dilatation

Favre-averaged Reynolds-stress tensor: $\tau_{ij} = \dfrac{-\overline{\rho u_i'' u_j''}}{\overline{\rho}}$

Turbulent kinetic energy per unit volume: $k = \dfrac{\overline{\rho u_i'' u_i''}}{2\overline{\rho}}$

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Terms to model

$$\bar{\rho}\frac{\partial k}{\partial t} + \bar{\rho}\tilde{u}_j \frac{\partial k}{\partial x_j} = \underbrace{\bar{\rho}\tau_{ij}\frac{\partial \tilde{u}_i}{\partial x_j}}_{P_k} - \underbrace{\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}}}_{\bar{\rho}\epsilon} + \underbrace{\frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j}}_{D} - \underbrace{\frac{\partial \overline{u_j'' P'}}{\partial x_j}}_{\Pi_t} - \underbrace{\frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j}}_{T_k} - \underbrace{\overline{u_i''}\frac{\partial \bar{P}}{\partial x_i}}_{\Pi_w} + \underbrace{\overline{P'\frac{\partial u_i''}{\partial x_i}}}_{\Pi_d}$$

Two eq. model may have own transport eq. (k-ε, k-ω, etc.)
Zero and one eq., modeled based on k etc.

Favre-averaged Reynolds-stress tensor: $\tau_{ij} = \frac{-\overline{\rho u_i'' u_j''}}{\bar{\rho}}$

Turbulent kinetic energy per unit volume: $k = \frac{\overline{\rho u_i'' u_i''}}{2\bar{\rho}}$

Decompositions

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Favre-Averaged Governing Equations - Terms to Model

From Mass, Momentum, and Energy

Favre-averaged Reynolds-stress tensor: $\tau_{ij} = \dfrac{-\overline{\rho u_i'' u_j''}}{\bar{\rho}}$

Can get k from this

Turbulent heat flux: $q_{Tj} = \overline{\rho u_j'' h''}$

Molecular diffusive flux of turbulent kinetic energy: $\overline{t_{ij} u_i''}$

Turbulent transport flux of turbulent kinetic energy: $-\dfrac{1}{2}\overline{\rho u_j'' u_i'' u_i''}$

<u>Good starting point for compressible-flow closure approximations</u>

NASA Langley Turbulence Modeling Resource:

- **Implementing Turbulence Models into the Compressible RANS Equations**
  - https://tmbwg.github.io/turbmodels/implementrans.html

Favre-Averaged Governing Equations - Terms to Model

Extra terms to consider from TKE equation

Pressure diffusion: $$\Pi_t = -\frac{\partial \overline{u_j'' P'}}{\partial x_j}$$

Pressure dilatation: $$\Pi_d = \overline{P' \frac{\partial u_i''}{\partial x_i}}$$

Pressure work: $$\Pi_w = -\overline{u_i''} \frac{\partial \overline{P}}{\partial x_i}$$

Viscous dissipation: $$\overline{\rho}\epsilon = -\overline{t_{ij} \frac{\partial u_i''}{\partial x_j}}$$

2-eqn. model, may get own transport equation

Developing contemporary models for these terms remains an active research area and is further motivated by recent interest in highly compressible and hypersonic flows

Compressible Flow Closure Approximations - Favre-averaged Reynolds-stress tensor

- Use an eddy viscosity and the Boussinesq approximation
- Relate the turbulent stress tensor to the mean strain rate tenson

$$\bar{\rho}\tau_{ij} = \overline{\rho u_i'' u_j''} \approx 2\mu_T \left(\widetilde{S_{ij}} - \frac{1}{3}\frac{\partial \widetilde{u_k}}{\partial x_k}\delta_{ij} \right) - \frac{2}{3}\bar{\rho} k \delta_{ij}$$

$$\widetilde{S_{ij}} = \frac{1}{2}\left(\frac{\partial \widetilde{u_i}}{\partial x_j} + \frac{\partial \widetilde{u_j}}{\partial x_i} \right)$$

Compressible Flow Closure Approximations - Turbulent Heat Flux

- Relate momentum and heat transfer (classical Reynolds analogy)
- Turbulent heat flux assumed proportional to mean temperature gradient
- Constant turbulent Prandtl number $\mathrm{Pr_T} \approx 0.9$ often assumed for boundary layers and $\mathrm{Pr_T} \approx 0.5$ for free shear layers, but can vary spatially

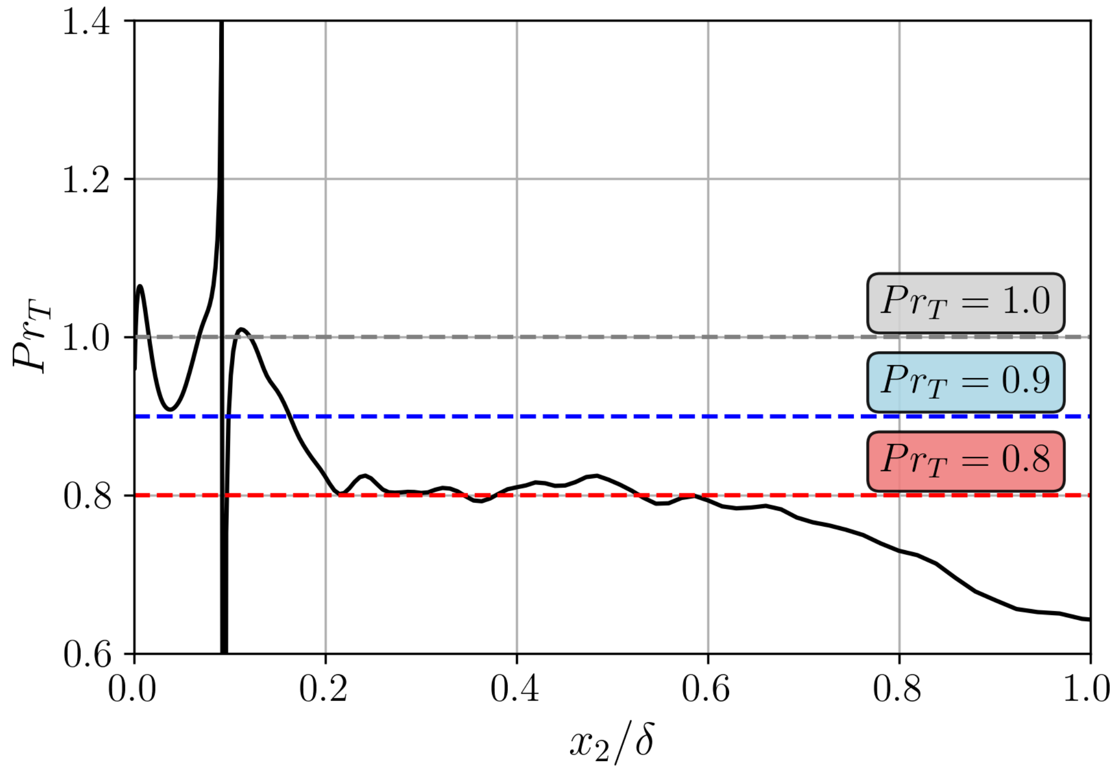
$$q_{Tj} = \overline{\rho u_j'' h''} \approx -\frac{\mu_T c_p}{Pr_T}\frac{\partial \widetilde{T}}{\partial x_j} = -\frac{\mu_T}{Pr_T}\frac{\partial \widetilde{h}}{\partial x_j}$$

Compressible Flow Closure Approximations - Turbulent Heat Flux

- It the turbulent Prandtl number constant?

Can compute it exactly from experimental or direct numerical simulation (DNS) data and see its variation across the boundary layer

$$Pr_T = \frac{\overline{\rho u_1'' u_2''}\,\frac{\partial \widetilde{T}}{\partial x_2}}{\overline{\rho T'' u_2''}\,\frac{\partial \widetilde{u_1}}{\partial x_2}}$$



M = 2.5, Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS

Compressible Flow Closure Approximations - Molecular Diffusion and Turbulent Transport of TKE

- In addition to advection, turbulent kinetic energy (TKE) can be transported via molecular diffusion and from the turbulent fluctuations themselves
- These two mechanisms are often modeled together

Molecular diffusive flux of TKE $\overline{t_{ij}u_i''}$

Turbulent transport flux of TKE $-\frac{1}{2}\overline{\rho u_j'' u_i'' u_i''}$

$$\overline{t_{ij}u_i''} - \frac{1}{2}\overline{\rho u_j'' u_i'' u_i''} \approx \left(\bar{\mu} + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}$$

- Coefficient σ_k associated with the modeling equation for k (Wilcox 2006)
- For zero-equation models and subsonic flows, both terms often neglected

Compressible Flow Closure Approximations - Pressure Diffusion of TKE

- The pressure diffusion term acts to transport turbulent kinetic energy (TKE) like the molecular diffusion and turbulent transport terms
- Pressure diffusion is often modeled/absorbed with the turbulent transport

Molecular diffusive flux of TKE $\overline{t_{ij}u_i''}$

Turbulent transport flux of TKE $-\frac{1}{2}\overline{\rho u_j'' u_i'' u_i''}$

Pressure transport flux of TKE $-\overline{u_j'' P'}$

$$\overline{t_{ij}u_i''} - \frac{1}{2}\overline{\rho u_j'' u_i'' u_i''} - \overline{u_j'' P'} \approx \left(\bar{\mu} + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}$$

- Pressure diffusion is implicitly ignored by lumping with turbulent transport

Compressible Flow Closure Approximations - Pressure Dilatation of TKE

- The pressure dilatation term has been shown to be small relative to the other TKE transport terms
- Pressure dilatation is often neglected for low Mach flows, but is non-negligible for high-speed flows (Sarkar 1992)
  - Pressure dilatation sometimes lumped into dilatational component of the viscous dissipation

$$\Pi_d = \overline{P' \frac{\partial u_i''}{\partial x_i}} \approx \alpha_2 \bar{\rho} \tau_{ij} \frac{\partial \widetilde{u}_i}{\partial x_j} M_t + \alpha_3 \bar{\rho} \epsilon M_t^2$$

$$\alpha_2 = 0.15 \quad \alpha_3 = 0.2 \quad M_t = \frac{\sqrt{2k}}{\widetilde{a}} \quad \widetilde{a} = \sqrt{\gamma R_{gas} \widetilde{T}}$$

Proposed model by Sarkar 1992

Compressible Flow Closure Approximations - Pressure Work

- Need to model the non-zero Favre fluctuation that appears in pressure work
- Favre-fluctuation of velocity is the "turbulent mass flux"

Pressure work: $\Pi_w = -\overline{u_i''}\frac{\partial \overline{P}}{\partial x_i}$

Turbulent mass flux: $\overline{u_i''} = \frac{-\overline{\rho' u_i'}}{\overline{\rho}}$

- Two example modeling forms:

$$-\overline{\rho' u_i'} \approx D_T \frac{\partial \overline{\rho}}{\partial x_i}$$

Proposed model taking gradient diffusion form (Zeman 1993, Ristorcelli 1993, Chassaing 2001)

$$\frac{\overline{\rho' u_i'}}{\overline{\rho}} \approx \frac{\overline{P' u_i'}}{\overline{P}} + \frac{\overline{T' u_i'}}{\overline{T}}$$

Linear link between pressure-velocity, temperature-velocity, and mass-fraction-velocity correlations via the equation of state (Chassaing 2001)

Compressible Flow Closure Approximations - Viscous Dissipation

$$\bar{\rho}\epsilon = \bar{\rho}\epsilon_s + \bar{\rho}\epsilon_d$$

- Dissipation can be split into solenoidal (ϵ_s) and dilatational (ϵ_d) components
  - See (Wilcox 2006) for assumptions and procedure

$$\bar{\rho}\epsilon = \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} = \frac{1}{2}\overline{t_{ij}\left(\frac{\partial u_i''}{\partial x_j} + \frac{\partial u_j''}{\partial x_i}\right)} = \overline{t_{ij}s_{ij}''}$$

$$\bar{\rho}\epsilon_s = \frac{\bar{\mu}}{\bar{\rho}}\overline{\rho\omega_i''\omega_i''}$$

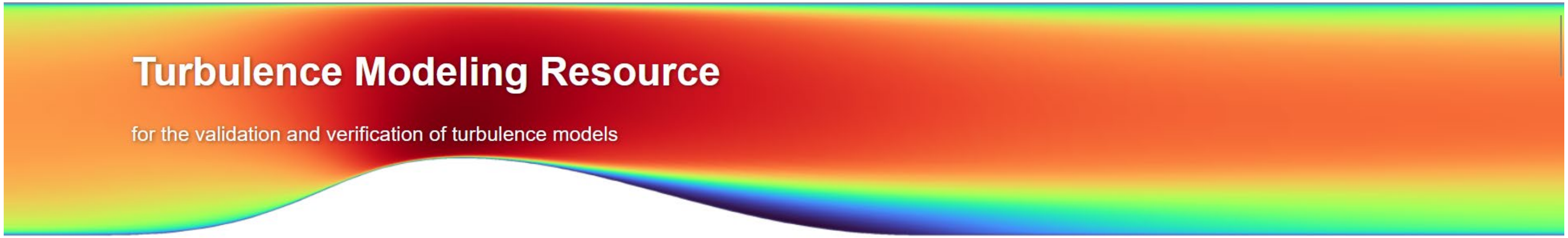
$$\bar{\rho}\epsilon_d = \frac{4}{3}\frac{\bar{\mu}}{\bar{\rho}}\overline{\rho\frac{\partial u_i''}{\partial x_i}\frac{\partial u_i''}{\partial x_i}}$$

- When modeling, the dissipation rate may be solved via its own transport equation (two-equation models like k − ϵ or k −ω), or modeled algebraically (e.g. as a function of k and turbulence length/time scale)
- ϵ_s may be treated as the primary dissipation variable and ϵ_d is modeled as a function of ϵ_s and the turbulent Mach number

Compressible Flow Closure Approximations - Compressible Turbulence Models

- Compressible turbulence modeling follows closely incompressible framework:
  - Two-equation models (e.g., $k - \epsilon$ or $k - \omega$)
  - One-equation models (e.g., Spalart-Allmaras with compressibility corrections)
  - Algebraic zero-equation models (e.g., Prandtl's mixing-length model)
- BUT models must be appropriately extended to account for compressibility:
  - Variable density
  - Dilatational contributions to dissipation
  - Pressure-velocity and temperature-velocity correlations

NASA Turbulence Modeling Resource: https://tmbwg.github.io/turbmodels/index.html



What is a Shock-Boundary Layer Interaction (SBLI)?

Sarah Kinney

- Occurs when a **shock wave interacts with a turbulent boundary layer.**
- **SBLIs present significant aerothermal challenges in high-speed flows:**
  - APGs can induce separation even at moderate shock strengths.
  - Shock impingement and reattachment drive localized peak heating zones

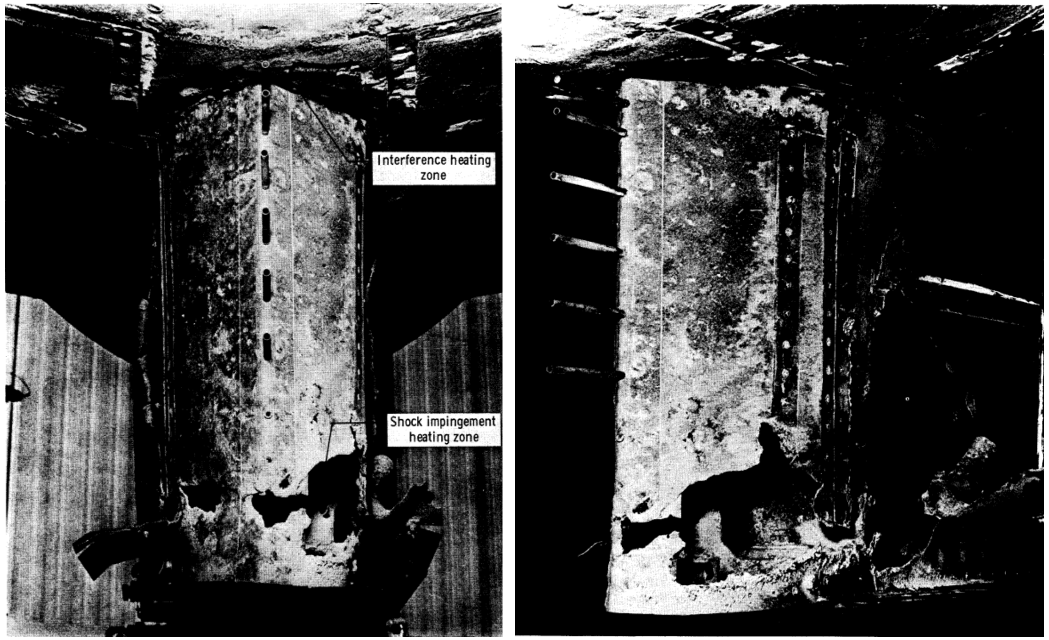


Figure 3. X-15 Flight Damage from Shock Impingement & Interference Heating [1]

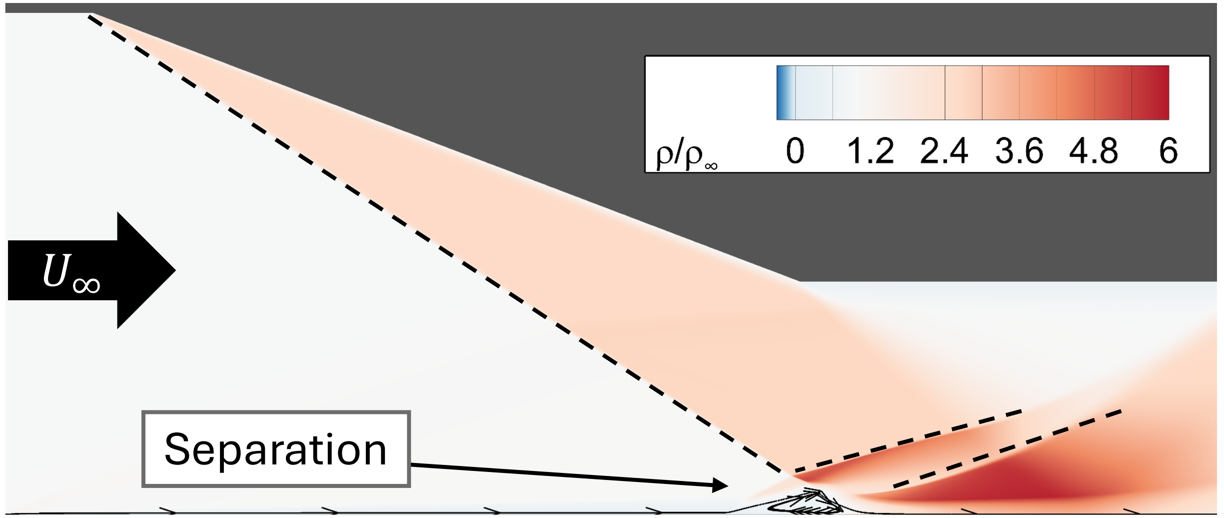


Figure 1. Impinging Shock Generator (ISG)

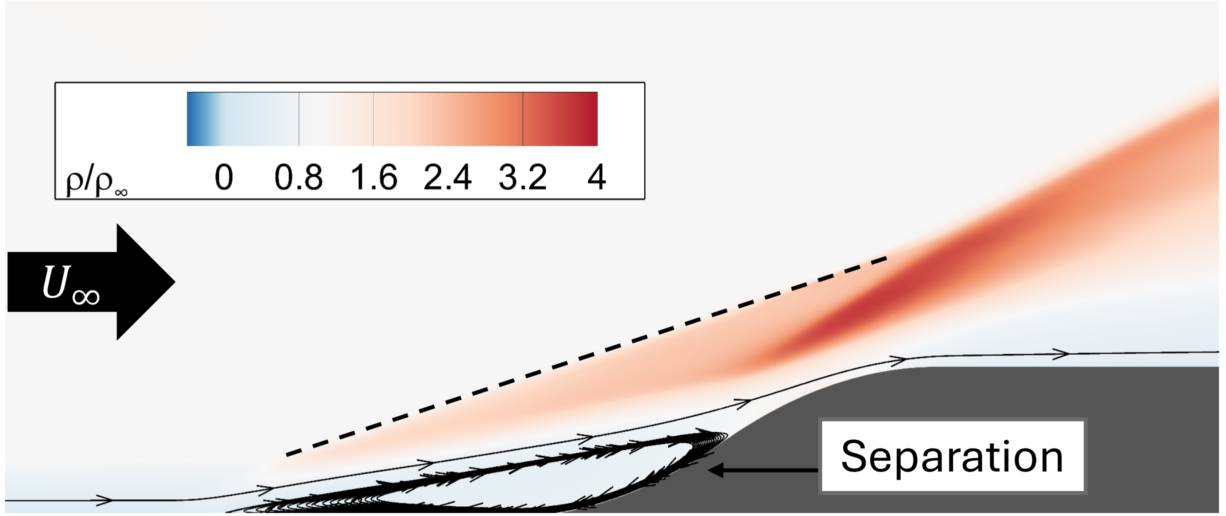


Figure 2. Forward-Facing Curved Wall (FFW)

Turbulence Kinetic Energy in SBLIs

$$\bar{\rho}\frac{\partial k}{\partial t}+\bar{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j}=P_k-\bar{\rho}\epsilon+D-\Pi_t-TT-\Pi_w+\Pi_d \quad (130)$$

- Notation note: Eq. 130 absorbs "M" in Fig. 4 into D, $\boldsymbol{\Pi_w}$, and $\boldsymbol{\rho\epsilon}$.
- TKE budget terms change significantly in presence of SBLIs.
- Terms that are often neglected in RANS models become non-negligible.
- Shear Stress Transport (SST) turbulence model captures general shape of production term, but other terms deviate significantly.
    - SST terms peak higher in BL due to SST's early prediction of separation location.

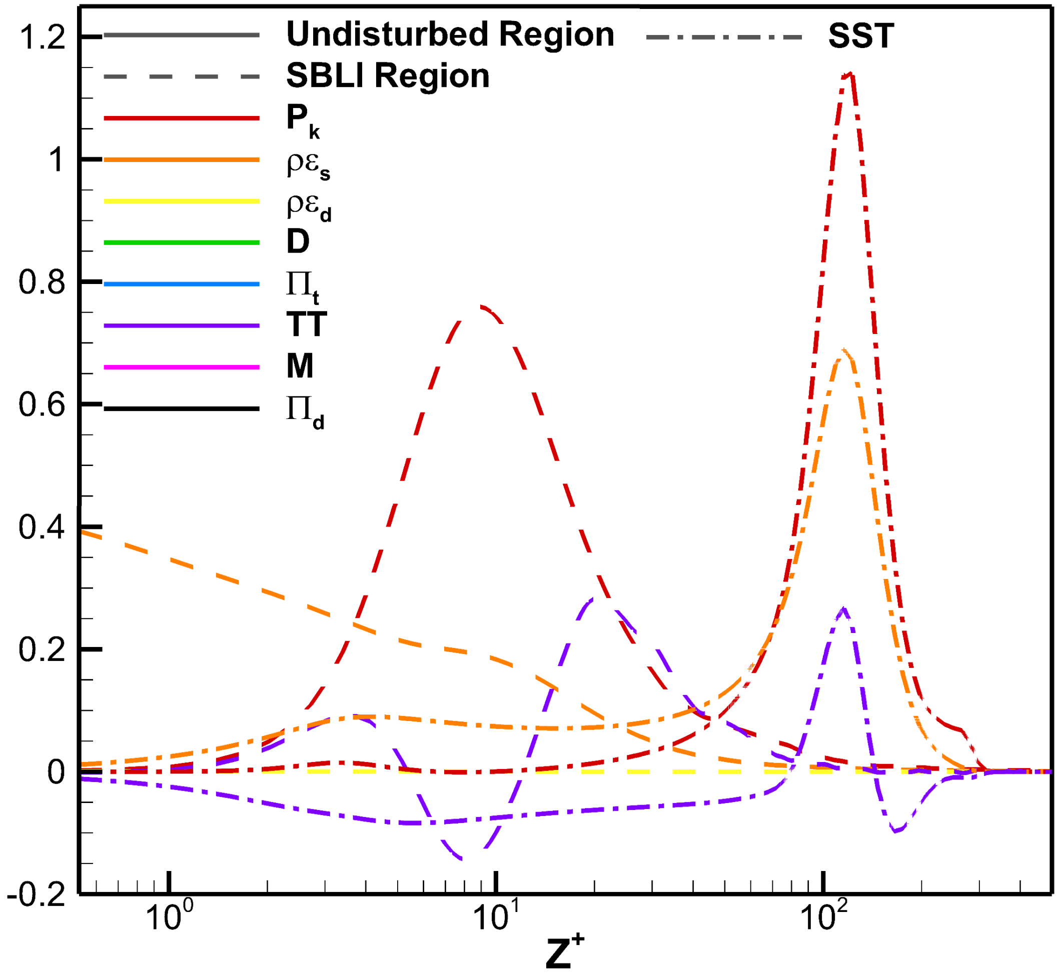


Figure 4. DNS vs SST TKE Budget for FFW, DNS from [2]

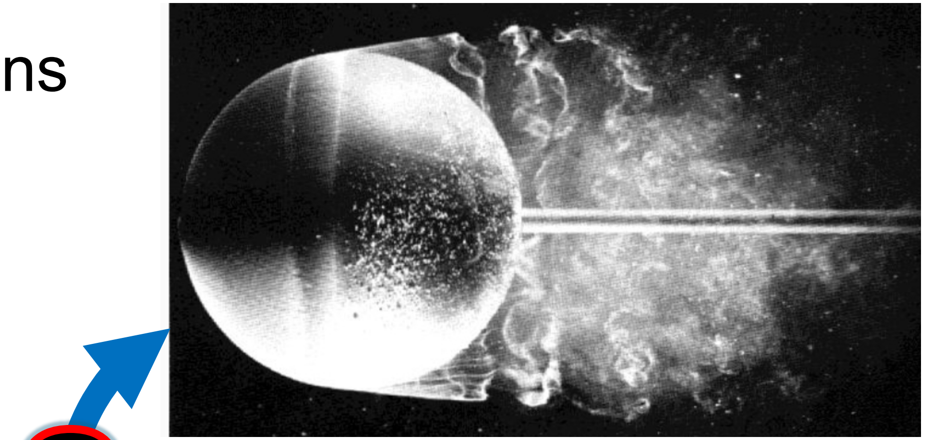
References

[1] Watts, Joe D.. “Flight Experience With Shock Impingement And Interference Heating On The X-15-2 Research Airplane”. 1968. NASA TM X-1669.

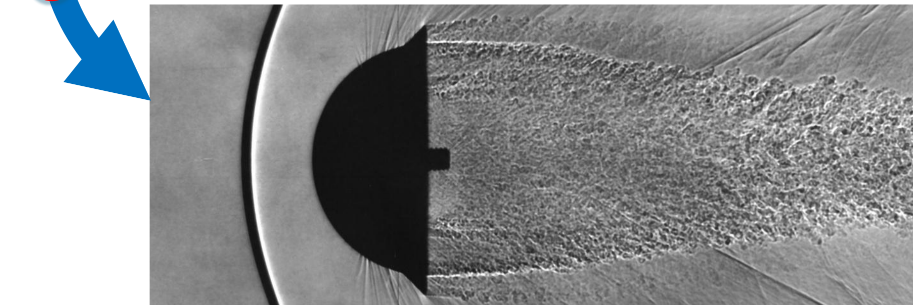
[2] Nicholson, Gary L., Duan, Lian. “Direct Numerical Simulation Database of High-Speed Flow Over Parameterized Curved Walls”. 2024. AIAA Journal.

Relationship to Incompressible Turbulence - How much incomp. theory holds at high Mach?

- Same governing equations, different assumptions and derivations (as seen in previous slides)
    - Still get conservation equations
    - Still get closure problem
    - Similar modeling framework (new terms)
- Concept of turbulence cascade still exists
    - Energy injected at large scales, energy cascade in inertial subrange, energy dissipation at small scales
- Near-wall flows still have a viscous sublayer, buffer layer, and log-layer. Law of the wall holds (sort of)



Incompressible flow: turbulence over sphere
Source: https://www.ams.jhu.edu/~eyink/TurbulenceII/



Compressible flow: turbulence over hemispherical hypersonic entry vehicle
Source: https://images.nasa.gov/details/ARC-1958-A-23753, NASA ID: ARC-1958-A-23753

Relationship to Incompressible Turbulence - Law of the Wall

- **Goal: Extend body of knowledge to from turbulence in incompressible flows to turbulence in compressible flows**
- Morkovin's hypothesis (Morkovin 1962):
  - Differences between compressible and incompressible turbulence can be accounted for by the mean property variations
    - Density
    - Viscosity
- Velocity and temperature scalings (transformations) have been developed to collapse the compressible statistics to their incompressible counterparts

Relationship to Incompressible Turbulence - Law of the Wall

- Incompressible boundary layer flows collapse to the law of the wall when the mean velocity and wall-normal distance are normalized in inner scaling:

$$y^+ = \frac{y u_\tau \overline{\rho_w}}{\overline{\mu_w}} \qquad u^+ = \frac{\overline{u}}{u_\tau}$$

Assumptions:

- Fully turbulent, high-Reynolds number ($Re_\tau \gg 1$), no pressure gradient
- Incompressible flow ρ and μ constant so $\rho_w = \rho = \overline{\rho}$ and $\mu = \mu_w = \overline{\mu}$.

- For compressible flow, scale the velocity profile and wall coordinate by the mean property variation → get an equivalent 'incompressible' form y_I and u_I. Then normalize and collapsed to the incompressible law of the wall:

$$y_I^+ = \frac{y_I u_\tau \overline{\rho_w}}{\overline{\mu_w}} \qquad u_I^+ = \frac{u_I}{u_\tau}$$

Relationship to Incompressible Turbulence - Law of the Wall

- Account for the mean property variation in terms of mapping functions f_I (wall distance) and g_I (mean velocity):

$$y_I = \int_0^y f_I \, dy \qquad u_I = \int_0^{\tilde{u}} g_I \, d\tilde{u}$$

Transformation	Acronym	Wall Distance f_I	Mean Velocity g_I
van Driest (1951)	VD	1	$(\rho^+)^{1/2}$
Zhang et al. (2012)	Z	1	$\frac{g_Z}{\mu^+}$
Trettel and Larsson (2016)	TL	$\frac{\partial}{\partial y}\left[\frac{y(\rho^+)^{1/2}}{\mu^+}\right]$	$\mu^+ \frac{\partial}{\partial y}\left[\frac{y(\rho^+)^{1/2}}{\mu^+}\right]$
Volpiani et al. (2020)	V	$\frac{(\rho^+)^{1/2}}{(\mu^+)^{3/2}}$	$\frac{(\rho^+)^{1/2}}{(\mu^+)^{1/2}}$
Griffin et al. (2021)	G	1	$S_t^+ \frac{dy^*}{d\tilde{u}^+}$
Hasan et al. (2023)	H	1	$\left(\frac{1+\kappa y^* D^c}{1+\kappa y^* D^i}\right)\left(1 - \frac{y}{\delta_v^*}\frac{d\delta_v^*}{dy}\right)(\rho^+)^{1/2}$

Relationship to Incompressible Turbulence - Law of the Wall

- Full length terms shown on table from previous slide:

$$\rho^+ = \frac{\overline{\rho}}{\overline{\rho_w}}$$

$$\mu^+ = \frac{\overline{\mu}}{\overline{\mu_w}}$$

$$S_Z = \frac{1}{\mu^+}\frac{\partial \widetilde{u}^+}{\partial y^+}$$

$$g_Z = \frac{-\frac{S_z}{2} + \left(\left(\frac{S_Z}{2}\right)^2 + 1 - \left(\mu^+\right)^2 S_Z\right)^{1/2}}{1 - \left(\mu^+\right)^2 S_Z}$$

$$S_{eq}^+ = \frac{1}{\mu^+}\frac{\partial \widetilde{u}^+}{\partial y^*}$$

$$S_{TL}^+ = \mu^+ \frac{\partial \widetilde{u}^+}{\partial y^*}$$

$$\tau^+ = \tau_{visc}^+ + \tau_{turb}^+$$

$$S_t^+ = \frac{\tau^+ S_{eq}^+}{\tau^+ + S_{eq}^+ - S_{TL}^+}$$

$$\kappa = 0.41$$

$$A^+ = 17$$

$$f(M_\tau) = 19.3 M_\tau$$

$$M_\tau = \frac{u_\tau}{\sqrt{\gamma R_{gas} T_w}}$$

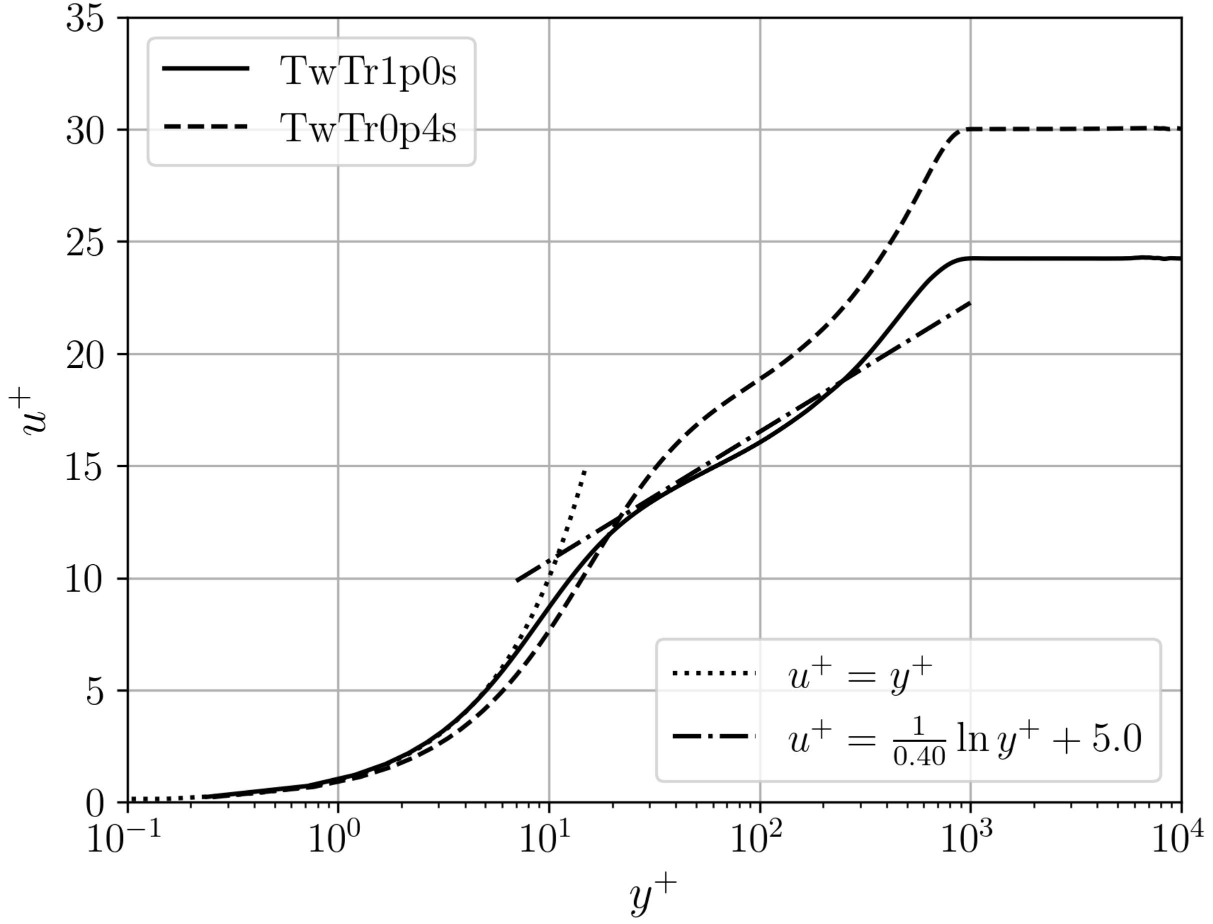
$$u_\tau^* = \sqrt{\frac{\tau_w}{\overline{\rho}(y)}}$$

$$\delta_v^* = \frac{\overline{\mu}(y)}{\overline{\rho}(y) u_\tau^*}$$

$$D^i = \left[1 - \exp\left(-\frac{y^*}{A^+}\right)\right]^2$$

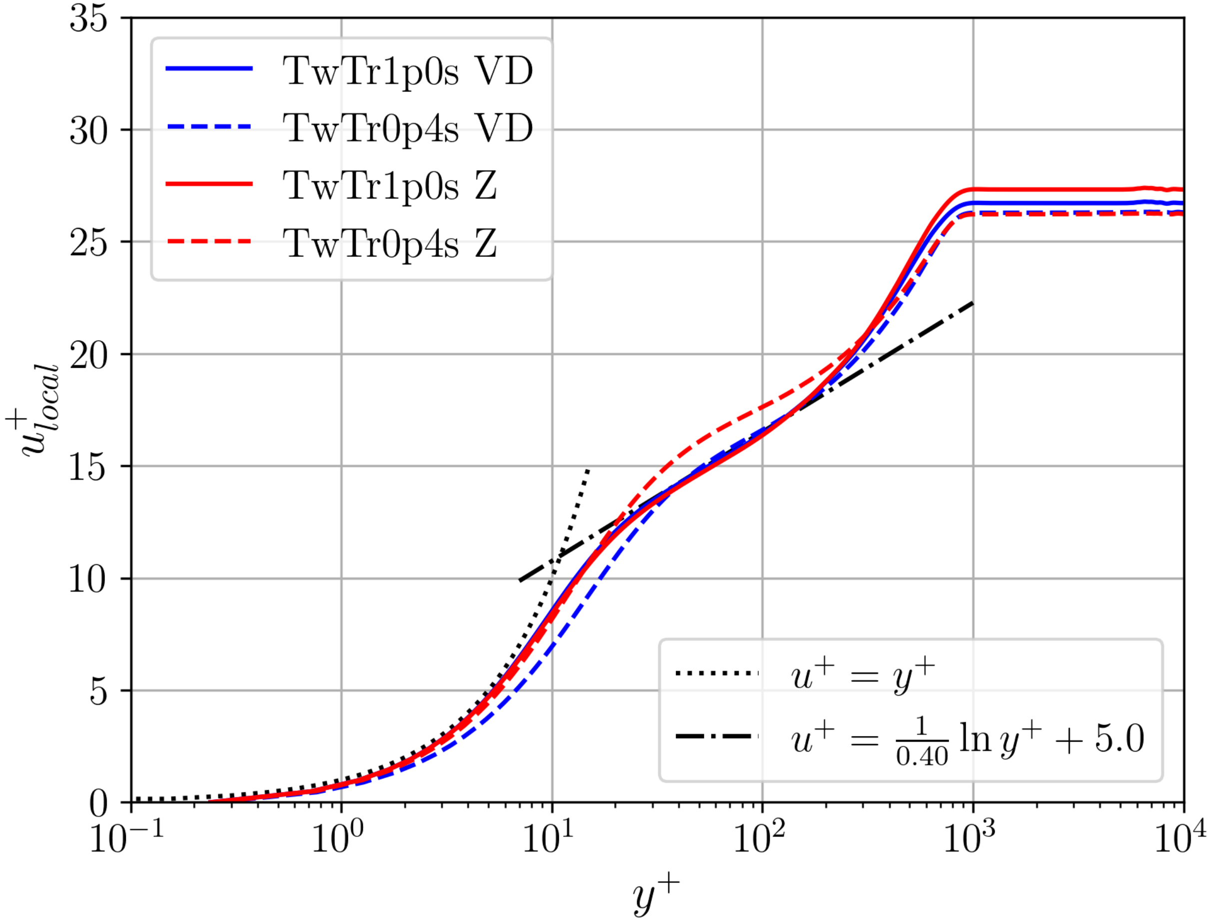
$$D^c = \left[1 - \exp\left(-\frac{y^*}{A^+ + f(M_\tau)}\right)\right]^2$$

Relationship to Incompressible Turbulence - Law of the Wall

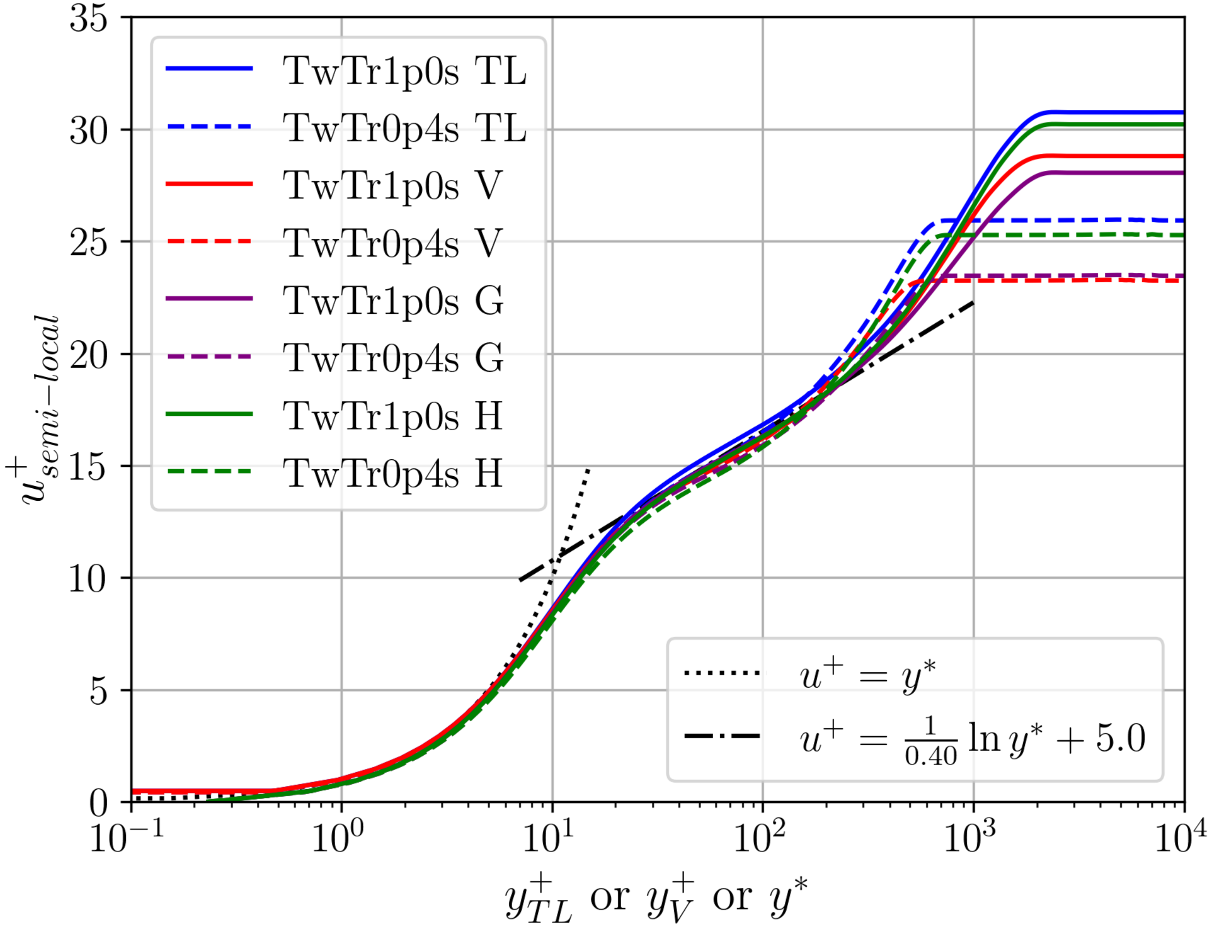


M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Naïve inner scaled velocity.

Relationship to Incompressible Turbulence - Law of the Wall



M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Local scaling velocity transformations.



M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Semi-local scaling velocity transformations.

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Relationship to Incompressible Turbulence - Law of the Wall: van Driest (1951)

- Compressibility effects are mainly due to mean density variation.
- If a log-law structure already exists, you can correct the velocity using a density weighting.
  - Assuming compressible wall-bounded turbulence obeys Prandtl's incompressible mixing length assumption $\left(\frac{\partial u_{VD}^{+}}{\partial y^{+}} = (\rho^{+})^{1/2} \frac{\partial \tilde{\mathrm{u}}^{+}}{\partial y^{+}}\right)$.
- Shown to work well for adiabatic walls, but breaks down when there is strong wall heating or cooling and large property variations (like viscosity or pressure gradients).

$$y^{+} = \frac{y_I u_\tau \overline{\rho_w}}{\overline{\mu_w}}$$

Local (wall) scaling (uses wall shear stress and wall properties)

Relationship to Incompressible Turbulence - Law of the Wall: Zhang et al. (2012)

- Sought a velocity transformation that is Mach number invariant.
- Replaced van Driest's mixing length assumption with the more general proposition of turbulence equilibrium (turbulence production and dissipation are approximately equal).
    - Usually only valid in the log-layer but they assume it for the entire inner layer.
- The result ends up being a transformation that is viscosity weighted.
- Like van Driest, also breaks down when there is strong wall heating or cooling (non-adiabatic).

Local (wall) scaling (uses wall shear stress and wall properties)

Relationship to Incompressible Turbulence - Law of the Wall: Trettel and Larsson (2016)

- Derived a transformation based on a “log-law” condition to match velocity gradient in the log-layer between the raw and transformed states, and a stress balance condition to match total stress (sum of viscous plus turbulent stress equals wall shear stress) between the raw and transformed states.
  - i.e. log-layer scaling and near-wall momentum conservation
- Derived the semi-local scaling (previously presented by Huang et al. 1995 and Coleman et al. 1995) and unified the scaling of the velocity, the Reynolds stresses, and the wall-normal coordinate.

$$y^* = \frac{y_I \sqrt{\tau_w / \rho(y)}}{\nu(y)} = y^+_{TL} = \frac{y_{I,TL} u_\tau \overline{\rho_w}}{\overline{\mu_w}}$$

Semi-local scaling (uses wall shear stress and local properties)

Relationship to Incompressible Turbulence - Law of the Wall: Volpiani et al. (2020)

- Considered mapping functions with power-law dependence on the density and viscosity ratios:

$$f_I = (\rho^+)^b (\mu^+)^{-a} \text{ and } g_I = (\rho^+)^b (\mu^+)^{1-a}$$

- Used DNS data to calibrate the values of the parameters a and b (goal to minimize the difference between the transformed mean velocity and incompressible reference).
  - Because this is a data-driven “fit”' the model may not work for all conditions or applications.

Semi-local scaling (uses wall shear stress and local properties)

Relationship to Incompressible Turbulence - Law of the Wall: Griffin et al. (2021)

- Presented a velocity transformation that is valid across the entire inner layer by using a total-stress-based balance to combine a viscous (near-wall) stress-based transformation like (Trettel and Larsson 2016) and a quasi-equilibrium-based transformation like (Zhang et al. 2012) in the log-layer.
  - Total shear stress is the sum of the viscous and Reynolds (turbulent) shear stresses
- The transformation reduces to the near-wall or log-layer portions only at the locations where the assumptions underlying each transformation are valid.
  - Only provide velocity transformation, no wall-distance transformation.
  - For plotting results, use the semi-local wall-normal coordinate.

Semi-local scaling (uses wall shear stress and local properties)

Relationship to Incompressible Turbulence - Law of the Wall: Hasan et al. (2023)

- All the previous transformations only accounted for mean property variation.
- This work accounts for both variable-property and intrinsic compressibility effects.
  - Extend the (Trettel and Larsson 2016) transformation for intrinsic compressibility effects.
- Identified the friction Mach number $M_\tau = \frac{u_\tau}{\sqrt{\gamma R_{gas} T_w}}$ as an important transformation parameter.

- Only provide velocity transformation, no wall-distance transformation.
- For plotting results, use the semi-local wall-normal coordinate.

Semi-local scaling (uses wall shear stress and local properties)

Transport Properties

- Molecules and atoms transfer energy via collisions.
- Collisions are governed by electrostatic forces between particles.
- Forces may be attractive or repulsive in nature.
- Attractive forces typically manifest as shear.
- Transfer of molecular kinetic energy during collisions manifests as thermal conduction.
- Mass diffusion simply arises from transfer of mass due to gradients in species concentrations (much like energy).

Transport property models are typically derived from kinetic theory of gases

Shrihari Ravichandran

Transport Properties

Sutherland's Model

- Assumes an idealized potential for molecular collisions
- Produces good agreement with experiments below ~2,500-3,000 K.
- Easy to evaluate within CFD codes!

$$\mu = \mu_{ref}\left(\frac{T}{T_{ref}}\right)^{3/2}\frac{T_{ref}+S}{T+S}$$

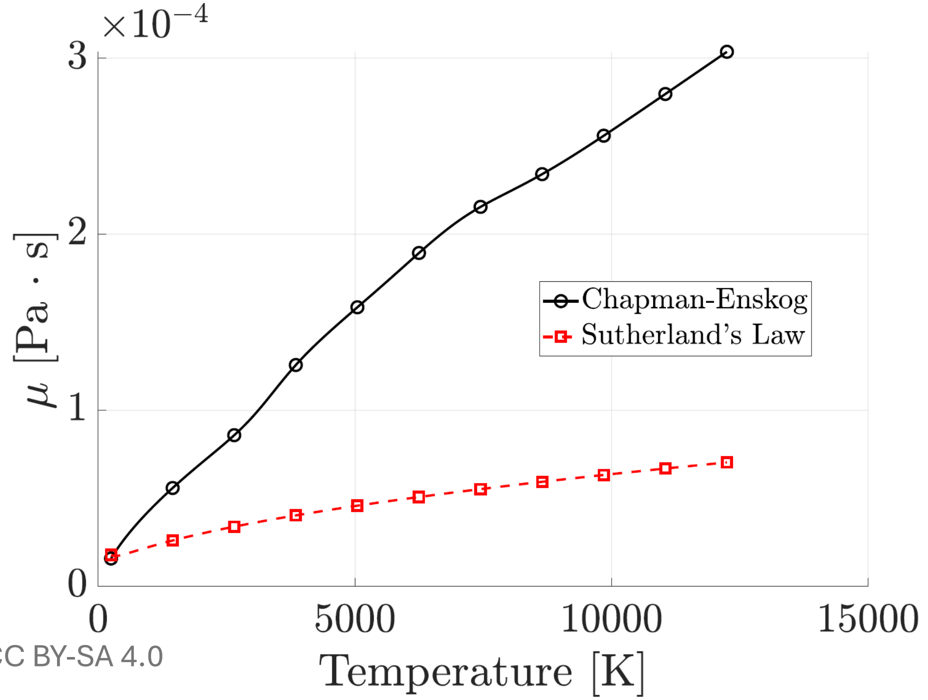
$$\eta = \mu c_P/Pr$$

Thermal conductivity also denoted with k or κ

- Idealized potential leads to errors at higher temperatures.
- Sutherland's Law also assumes thermally and calorically perfect gas i.e. chemistry cannot *strictly* be accommodated.

Chapman-Enskog Theory

- More rigorous approach that involves solving Boltzmann equations under some assumptions.
- Room for a variety of interaction potentials, from empirical to high-fidelity
- **More details: *Non-Equilibrium Reacting Gas Flows*, Nagnibeda and Kustova, Springer, 2009.**



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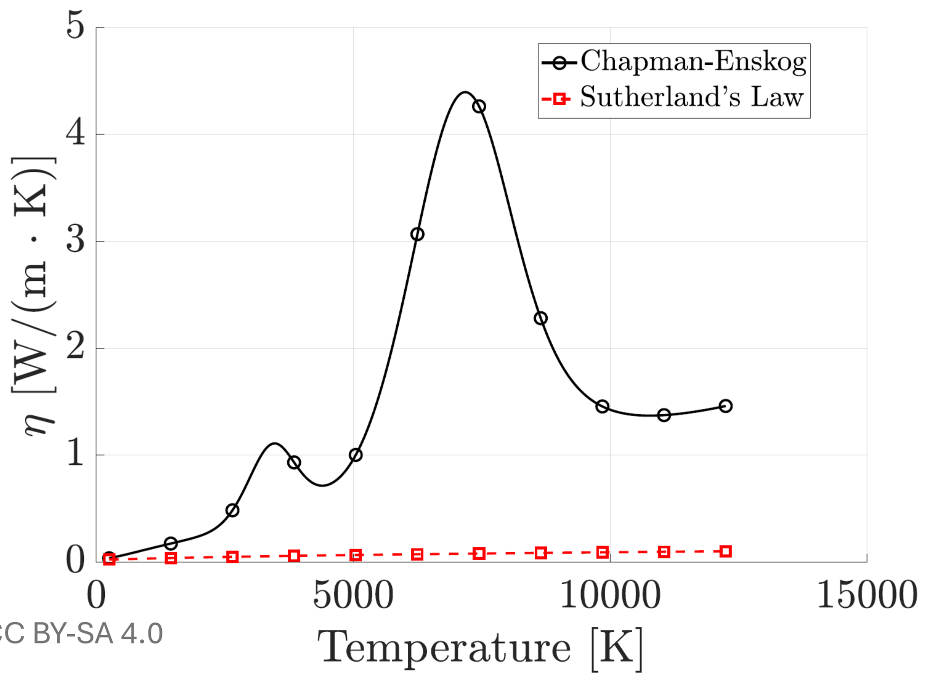
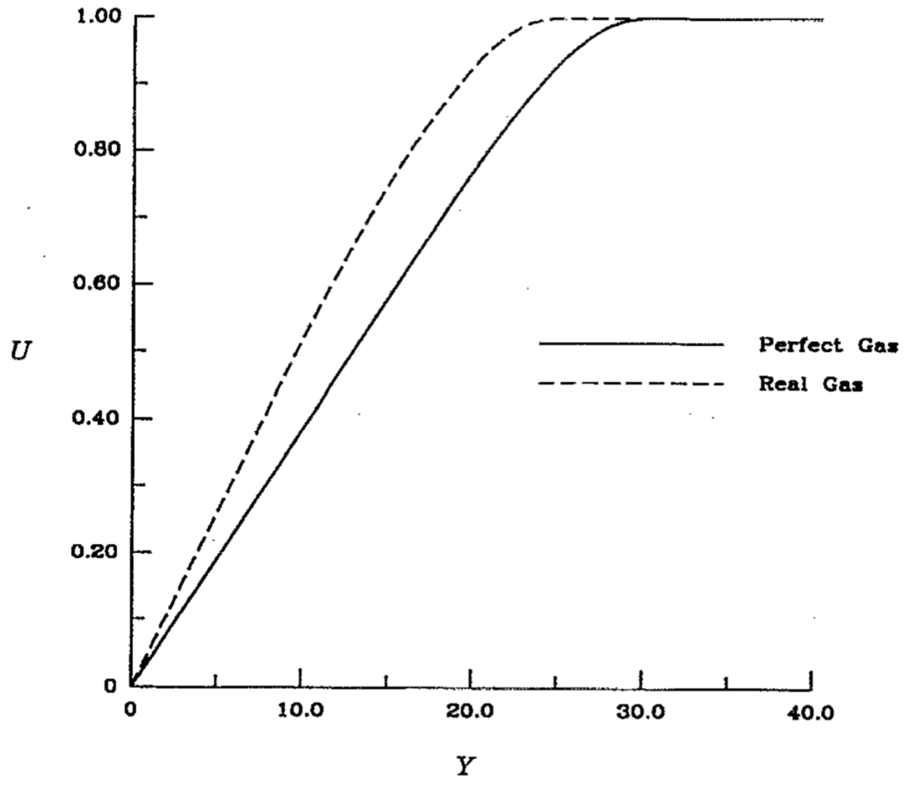
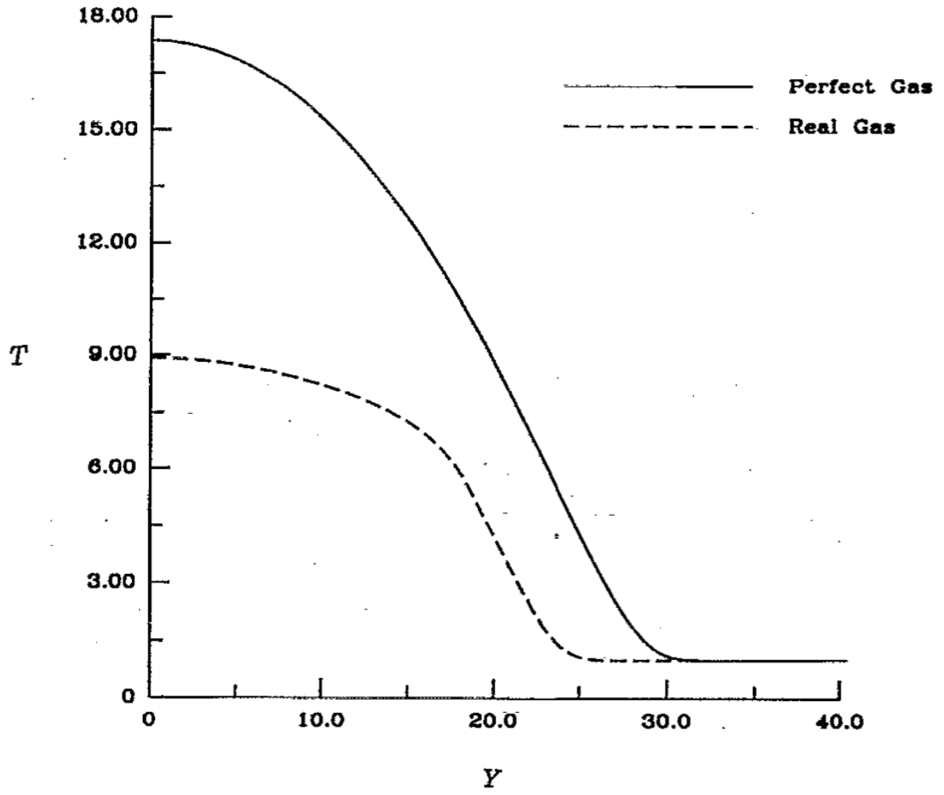


Effect of transport properties

- Study from **Malik and Anderson (PoF, 1991)** shows significant differences between boundary layer predictions that use Sutherland's Law and Chapman-Enskog-based methods.
- Study also shows major different in boundary layer stability calculations.



Normalized velocity vs wall-normal coordinate



Normalized temperature vs wall-normal coordinate

Energy/Temperature Considerations - Temperature Transformations

- The thermal (temperature) boundary layer can also be scaled in inner units
  - Collapse seen when looking at inner scaled temperature difference

$$\widetilde{\theta} = \widetilde{T_w} - \widetilde{T} \quad \text{or} \quad \overline{\theta} = \overline{T_w} - \overline{T}$$

- A friction temperature analogous to friction velocity is based on wall heat flux

$$\theta_\tau = \frac{\overline{q_w}}{\overline{\rho_w} c_p u_\tau} \qquad \theta^+ = \frac{\overline{\theta}}{\theta_\tau}$$

- Challenges for compressible flows:
  1. This scaling does not account for mean property variations
  2. The friction temperature normalization is undefined for adiabatic flow that has zero wall heat flux

Energy/Temperature Considerations - Temperature Transformations

- Analogous to the viscous sublayer, a thermal sublayer can be defined
  - Also known as the conductive or molecular transport sublayer because near the wall the molecular conduction dominates the heat transfer
  - It depends on the Prandtl number at the wall

$$\theta^+ = Pr_w y^+ \quad \text{or} \quad \theta^+ = Pr_w y^*$$

- Near the velocity boundary layer's log-layer, the logarithmic scaling of the mean velocity and the Reynolds analogy suggest a logarithmic scaling of the temperature

$$\theta^+ = \frac{1}{\kappa_T} \ln y^+ + B(Pr) \quad \text{or} \quad \theta^+ = \frac{1}{\kappa_T} \ln y^* + B(Pr)$$

$$1/\kappa_T \approx Pr_t/\kappa \longrightarrow \kappa_T \approx 0.47 \qquad B(Pr) \approx 3 - 5$$

Energy/Temperature Considerations - Temperature Transformations

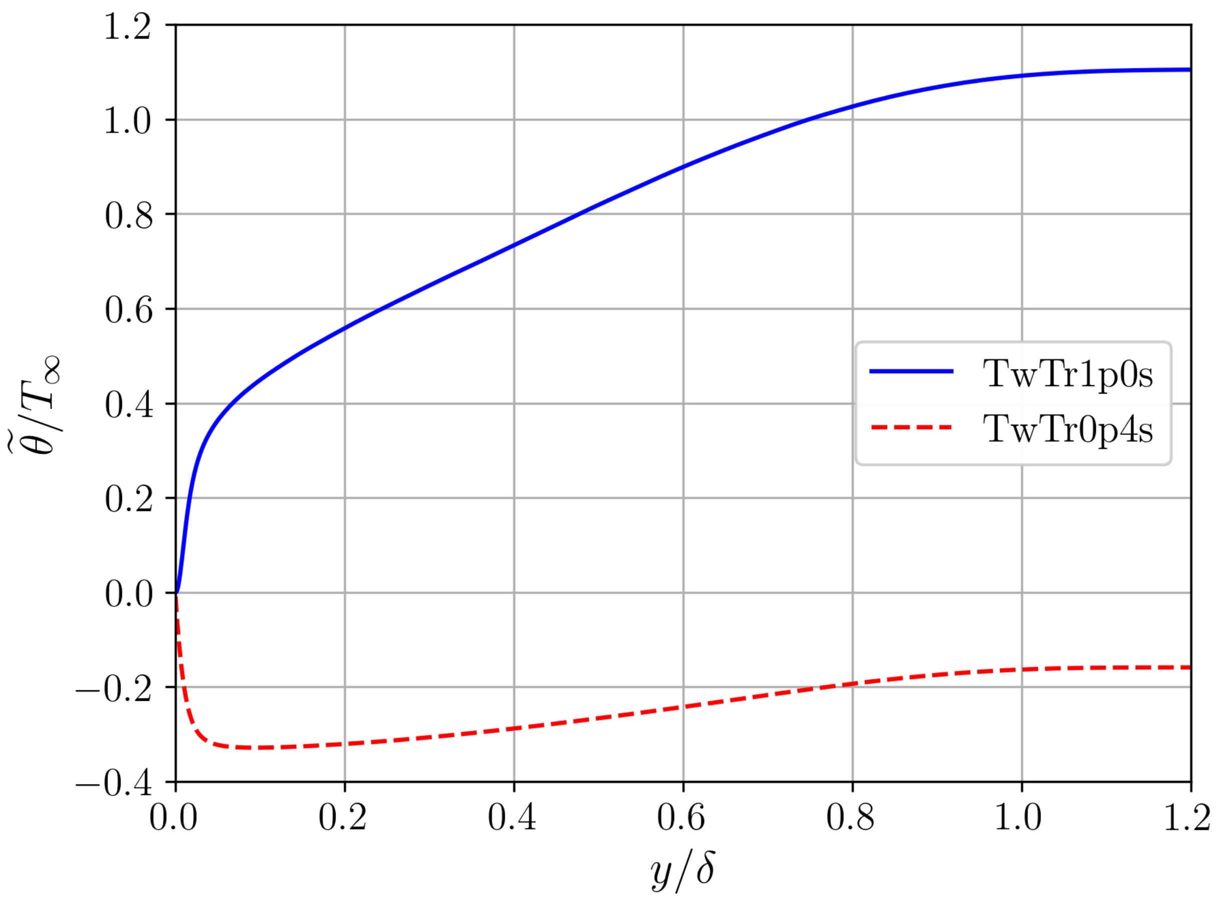
- Kader's relation (Kader 1981) attempts to combine the sublayer and log-layer regions into one function

$$\theta^+ = Pry^+ \exp\left(-\Gamma\right) + \left\{2.12 \ln\left[(1+y^+)\frac{2.5(2-y/\delta)}{1+4(1-y/\delta)^2}\right] + B(Pr)\right\} \exp\left(-1/\Gamma\right)$$

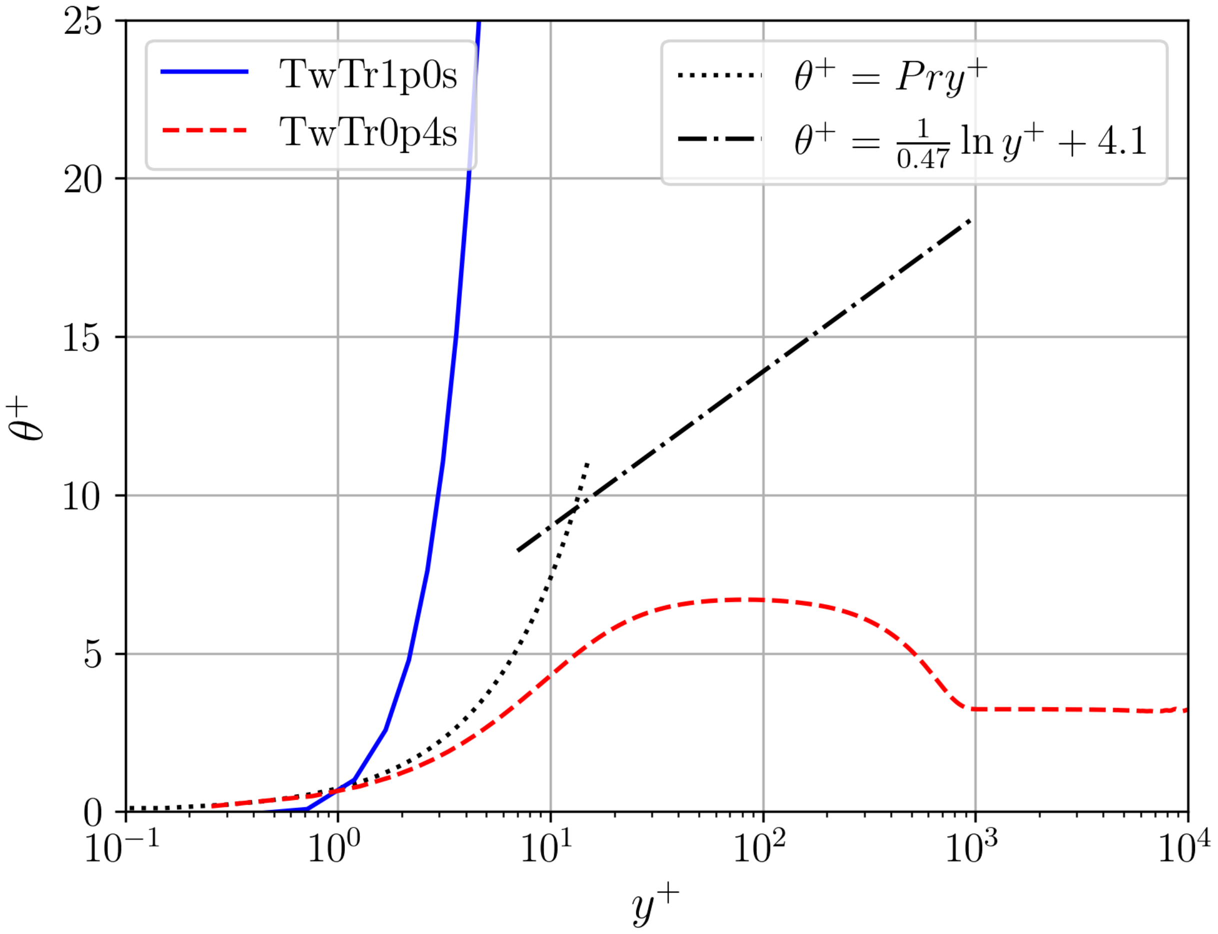
$$B(Pr) = (3.85Pr^{1/3} - 1.3)^2 + 2.12 \ln Pr \qquad \Gamma = \frac{10^{-2}(Pry^+)^4}{1+5Pr^3y^+}$$

- Extensions to Kader's relation (Lee et al. 2014) have been made to include local variations, making use of semi-local values

Energy/Temperature Considerations - Temperature Transformations

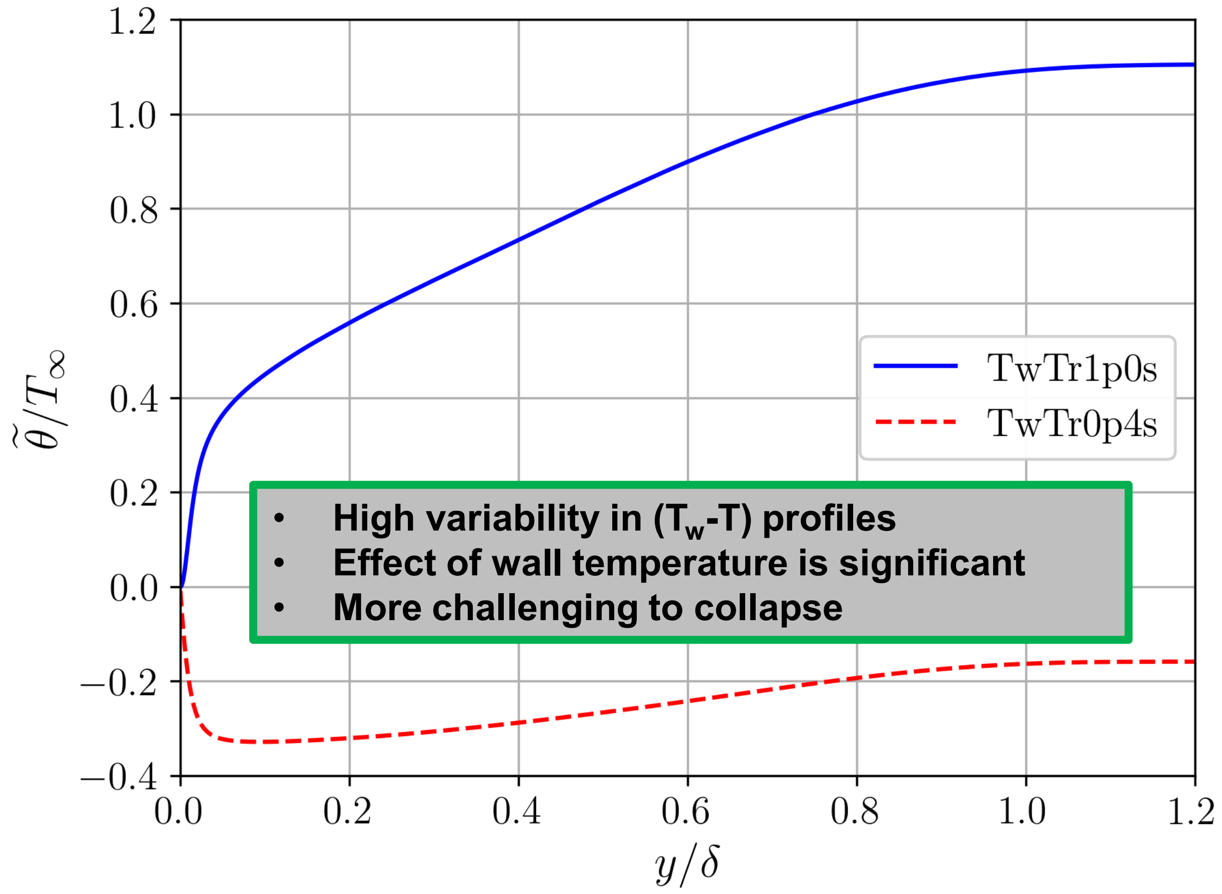


M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Outer scaled temperature difference.



M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Naïve inner scaled temperature.

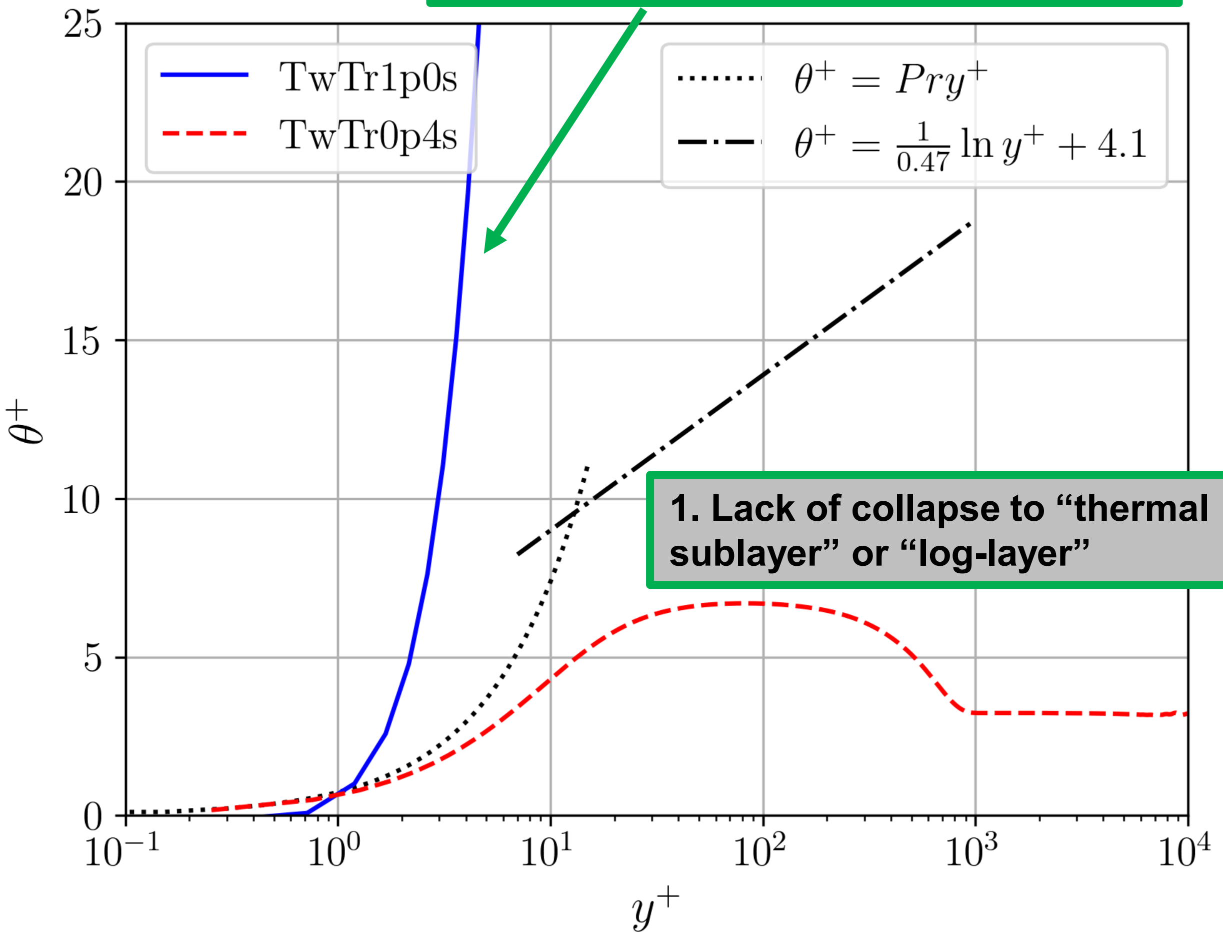
Energy/Temperature Considerations - Temperature Transformations



- High variability in (T_w-T) profiles
- Effect of wall temperature is significant
- More challenging to collapse

M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Outer scaled temperature difference.

2. Adiabatic wall case shoots to infinity because normalizing by zero heat flux!



1. Lack of collapse to "thermal sublayer" or "log-layer"

M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Naïve inner scaled temperature.

Energy/Temperature Considerations - Temperature Transformations

Solutions:

1. Mimic velocity and use an equivalent incompressible mapping to account for mean property variation

$$\theta^+ = \theta_I/\theta_\tau \longrightarrow \theta_I = \int_0^{\widetilde{\theta}} h_I \, d\widetilde{\theta} \quad \text{or} \quad \theta^+_{trans} = \int_0^{\theta^+} h_I \, d\theta^+$$

2. Define a friction temperature that accounts for the diffusive flux from the Favre-averaged energy equation in addition to the wall heat flux
   - Makes temperature transformation valid for both isothermal and adiabatic walls
   - This is active research and was only recently proposed (Chen et al. 2022)

$$\theta^*_{\tau,c} = \frac{\overline{q_w} + \overline{q}}{\overline{\rho} c_p u^*_\tau} \qquad \theta^+_{trans,c} = \int_0^{\widetilde{\theta}} h_I \, d\widetilde{\theta}$$

Energy/Temperature Considerations - Temperature Transformations

Transformation	Acronym	Wall Distance f_I	Mean Temperature h_I
van Driest - type Patel et al. (2017)	VD	1	$(\rho^+)^{1/2}$
semi-local - type Patel et al. (2017)	SL	1	$(\rho^+)^{1/2}\left[1+\frac{y}{Re_\tau^*}\frac{dRe_\tau^*}{dy}\right]$
van Driest - type Chen et al. (2022)	VDc	1	$(\theta_{\tau,c}^*)^{-1}$
semi-local - type Chen et al. (2022)	SLc	1	$(\theta_{\tau,c}^*)^{-1}\left[1+\frac{y}{Re_\tau^*}\frac{dRe_\tau^*}{dy}\right]$

$$\rho^+ = \frac{\overline{\rho}}{\overline{\rho_w}}$$

$$Re_\tau^* = Re_\tau\sqrt{\frac{\overline{\rho}}{\overline{\rho_w}}}\frac{\overline{\mu_w}}{\overline{\mu}}$$

$$u_\tau^* = \sqrt{\frac{\overline{\tau_w}}{\overline{\rho}}}$$

$$\theta_\tau^* = \frac{\overline{q_w}}{\overline{\rho} c_p u_\tau^*}$$

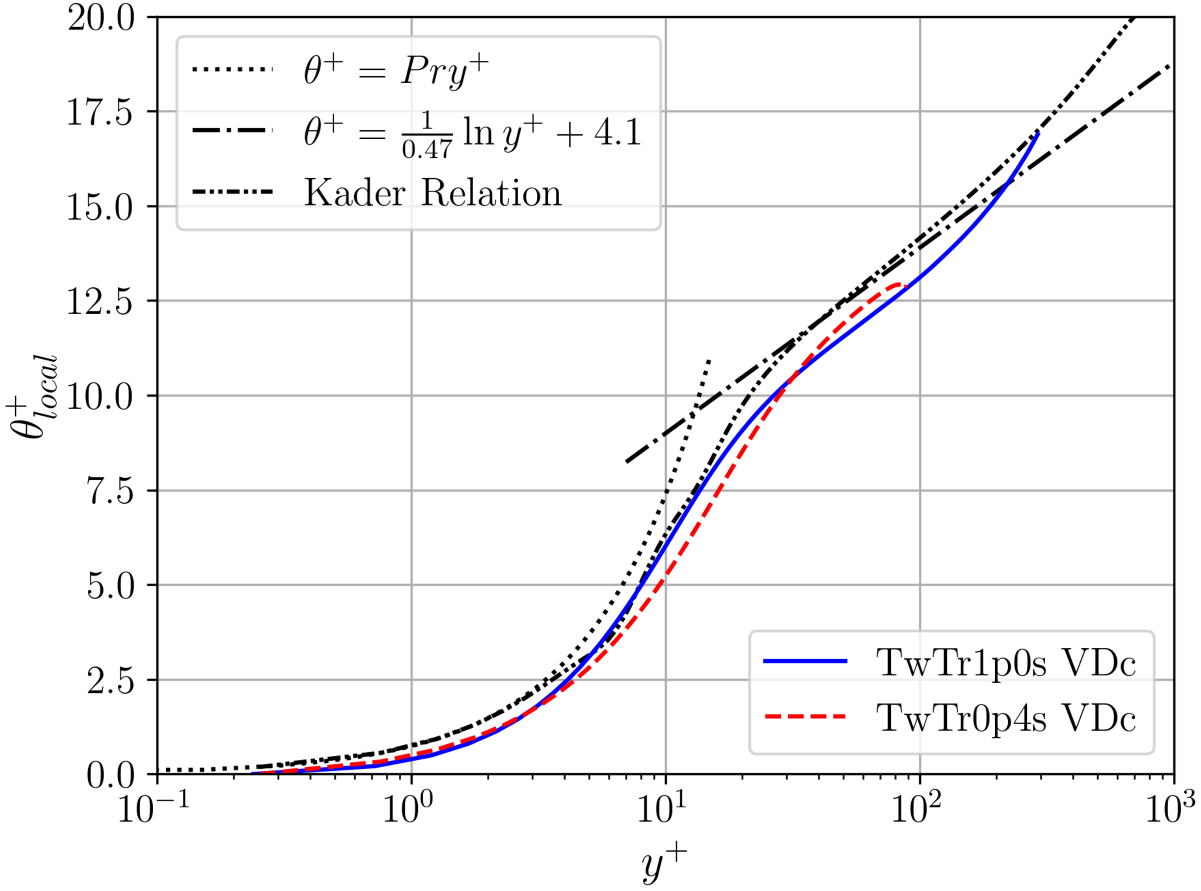
$$\theta_{\tau,c}^* = \frac{\overline{q_w}+\overline{q}}{\overline{\rho} c_p u_\tau^*}$$

- For cold walls you can get a negative temperature difference and a negative wall heat flux $q_w = -\kappa\frac{\partial T}{\partial y}$
- Ensuring consistent sign convention is necessary to for the negatives to cancel in the normalization

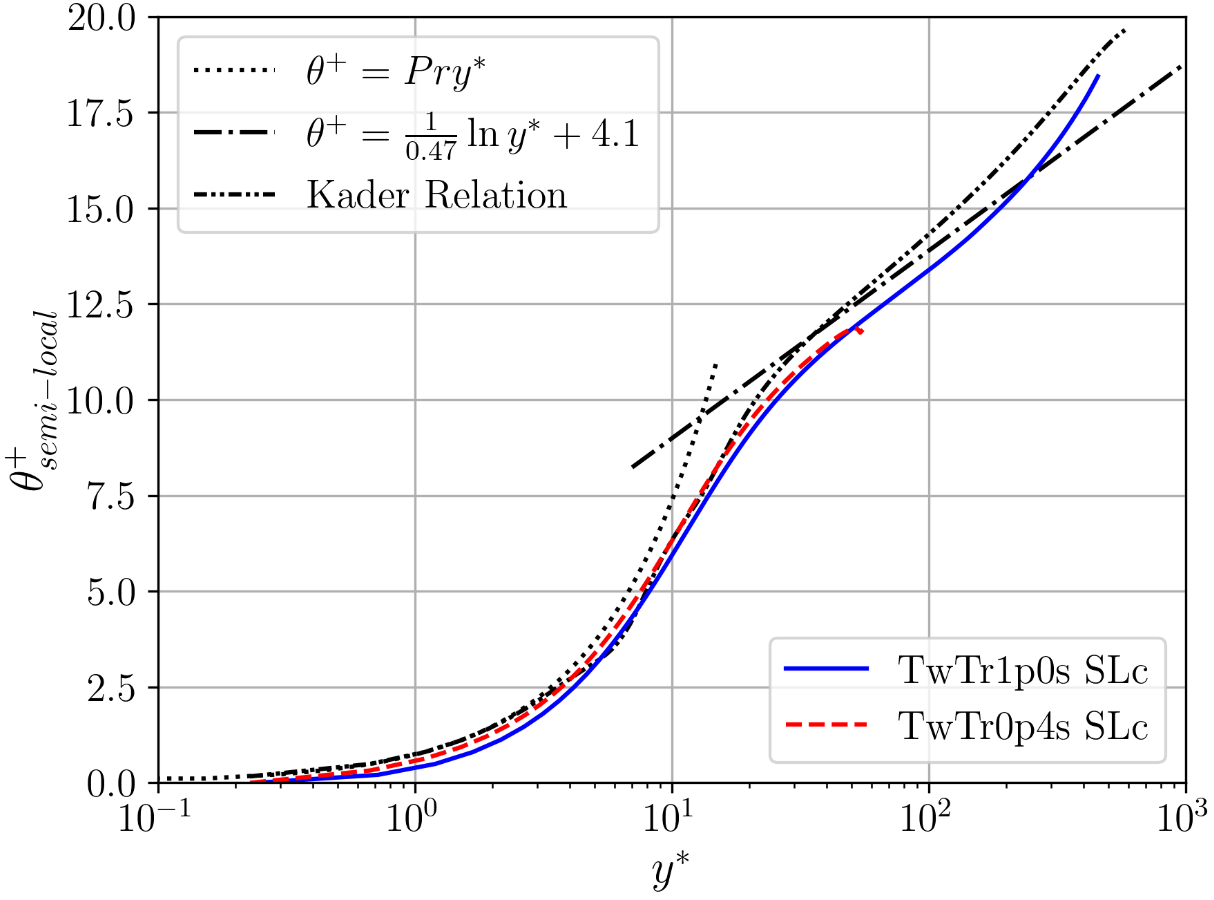
$$\overline{q} = \overline{t_{i2}u_i} - \overline{\rho v'' u_i''}\widetilde{u_i} - \frac{\overline{\rho v'' u_i'' u_i''}}{2}$$

Energy/Temperature Considerations - Temperature Transformations

θ^+ profiles truncated shortly after reaching log-layer because large variation in $T_w - T_\infty$, and $q_w + \overline{q}$ may be zero (or sign change) near outer layer



M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Local scaling temperature transformations. Corrected form to permit adiabatic wall data.



M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Semi-local scaled temperature transformations. Corrected form to permit adiabatic wall data.

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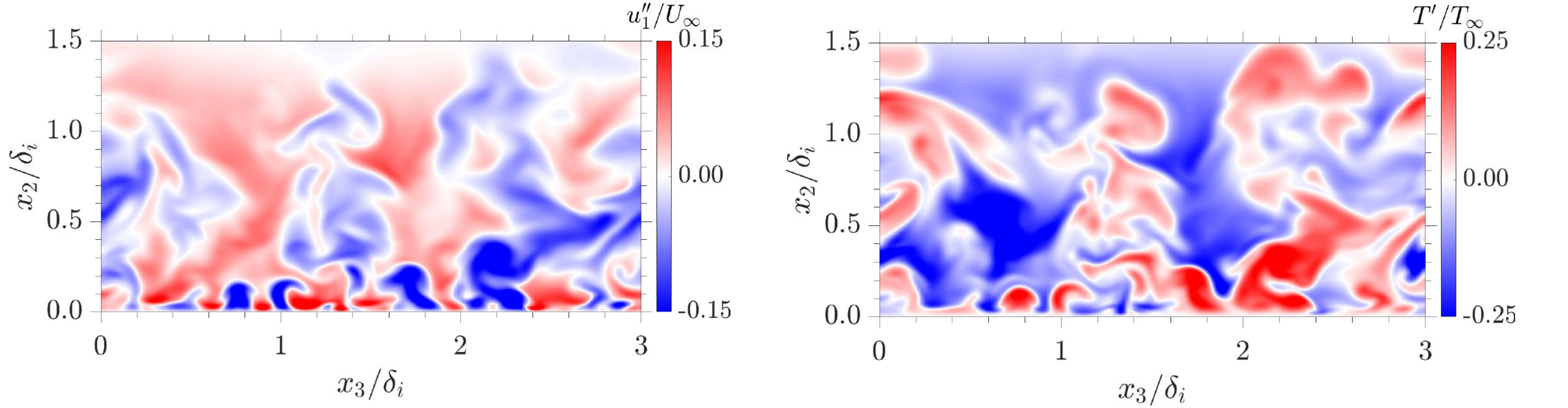
Energy/Temperature Considerations - Velocity-Temperature Relationships

- It has been known for over 100 years that there is a relationship between the streamwise velocity and temperature
- Analogies exist between total enthalpy and streamwise velocity, or skin friction coefficient and Stanton number
- This relationship is seen in laminar flow, the mean of turbulent flow, and even the fluctuations in turbulent flow



M = 2.5, Tw/Tr = 1.0, zero-pressure gradient flat plate turbulent boundary layer DNS. Comparison of streamwise velocity and temperature fluctuations.

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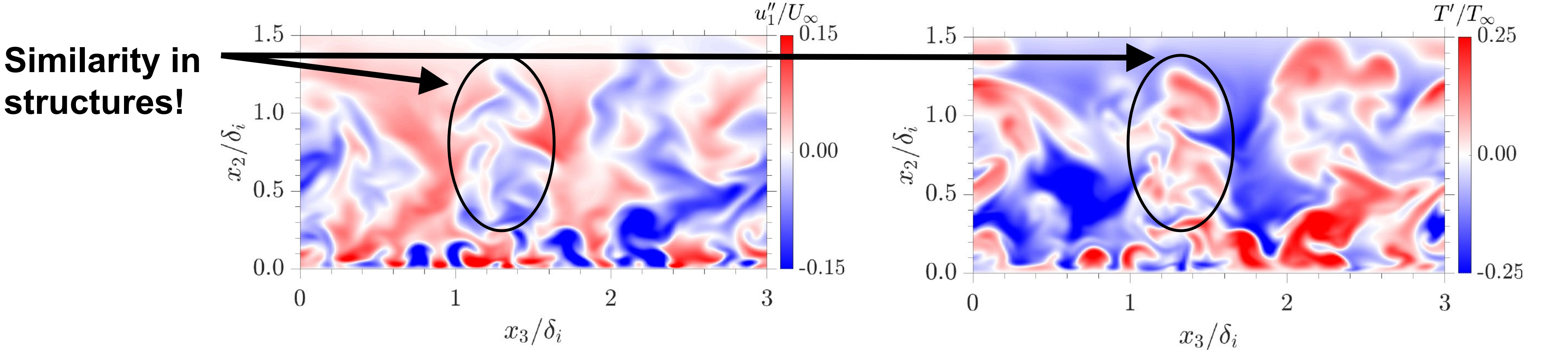


M = 2.5, Tw/Tr = 1.0, zero-pressure gradient flat plate turbulent boundary layer DNS. Comparison of streamwise velocity and temperature fluctuations.

Energy/Temperature Considerations - Velocity-Temperature Relationships (Mean)

- Busemann (1931) and Crocco (1932) showed temperature is proportional to the square of the streamwise velocity

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \frac{T_{c,\infty} - T_w}{T_\infty}\frac{u_1}{U_\infty} + \frac{T_\infty - T_{c,\infty}}{T_\infty}\left(\frac{u_1}{U_\infty}\right)^2$$

$$T_{c,\infty} = T_\infty + c\frac{U_\infty^2}{2c_p}$$

- Walz (1969) improved the Crocco-Busemann relation by setting c to be the recovery factor r

$$\frac{T}{T_\infty} = 1 + \frac{T_r - T_w}{T_\infty}\left(\frac{u_1}{U_\infty} - 1\right) + r\frac{\gamma - 1}{2}M_\infty^2\left(1 - \left(\frac{u_1}{U_\infty}\right)^2\right)$$

$$T_r = T_\infty\left(1 + r\frac{\gamma - 1}{2}M_\infty^2\right)$$

$$T_r = T_\infty + r\frac{U_\infty^2}{2c_p}$$

Energy/Temperature Considerations - Velocity-Temperature Relationships (Mean)

- To account for the effects where $Pr \neq 1$ and diabatic walls, Zhang et al. (2014) developed a generalized Reynolds analogy (GRA)
- Generalized analogy between the total enthalpy and streamwise velocity

$$\overline{H_g} - \overline{H_w} = U_w \overline{u_1} \qquad H_g = c_p T + r_g u_1^2/2 \qquad U_w = -Pr \overline{q_w}/\overline{\tau_w}$$

- Quadratic form:

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \frac{T_r - T_w}{T_\infty} f\left(\frac{u_1}{U_\infty}\right) + \frac{T_\infty - T_r}{T_\infty}\left(\frac{u_1}{U_\infty}\right)^2$$

Key assumption

$$\overline{Pr_e} \approx 1$$

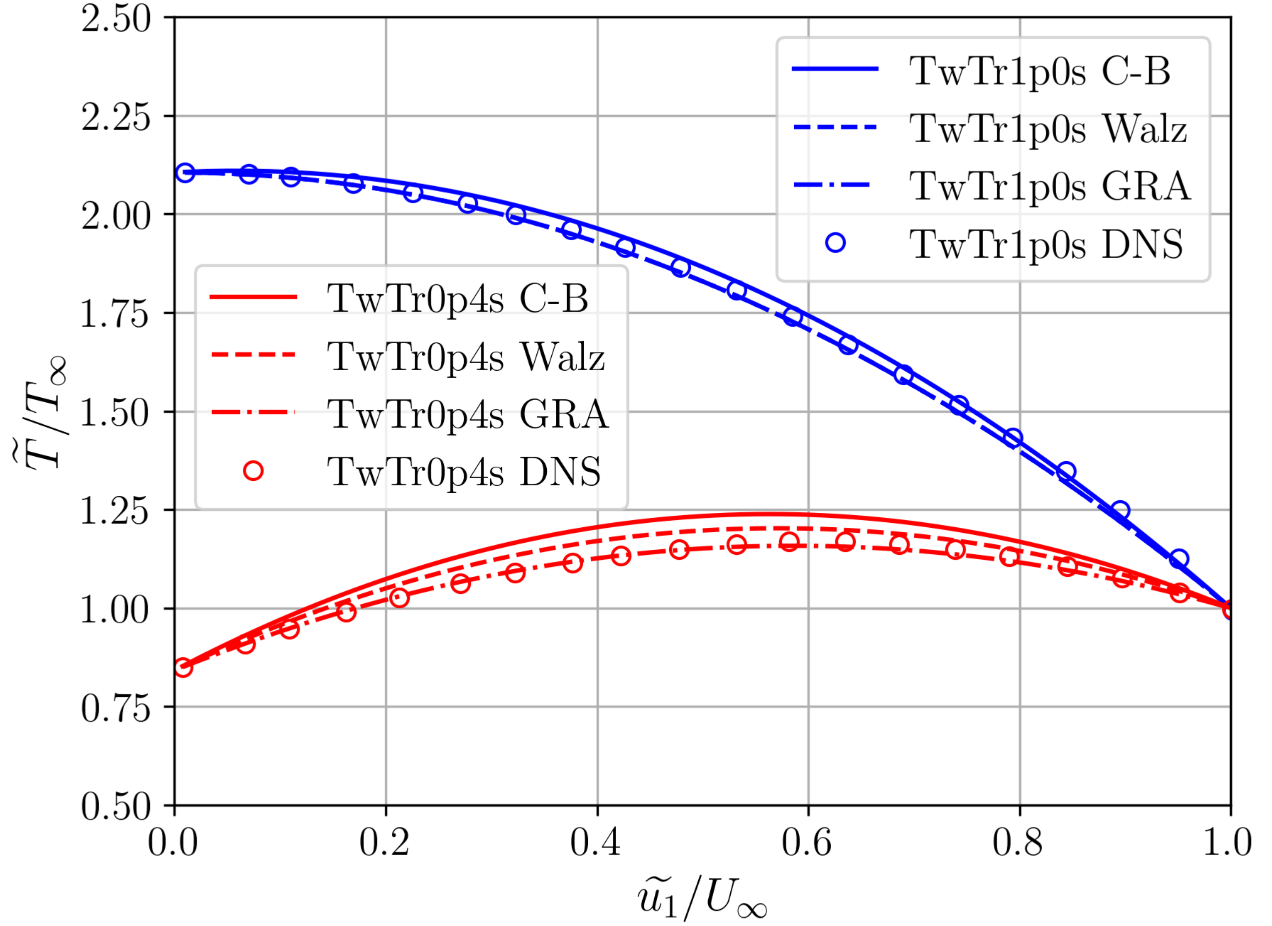
$$f\left(\frac{u_1}{U_\infty}\right) = (1 - sPr)\left(\frac{u_1}{U_\infty}\right)^2 + sPr\left(\frac{u_1}{U_\infty}\right)$$

$$r_g = \frac{T_w - T_\infty}{U_\infty^2/(2c_p)} - \frac{2Pr}{U_\infty}\frac{q_w}{\tau_w} = r\left[sPr + (1 - sPr)\Theta\right] \quad (\text{if } \overline{Pr_e} = 1)$$

If Reynolds analogy holds, Reynolds analogy factor will be 1

$$s \equiv \frac{2C_h}{C_f} = \frac{q_w U_\infty}{\tau_w c_p (T_w - T_r)} \qquad \Theta = \frac{T_w - T_\infty}{T_r - T_\infty} \qquad C_h = \frac{q_w}{\rho_\infty U_\infty c_p (T_w - T_r)} \qquad C_f = \frac{2\tau_w}{\rho_\infty U_\infty^2}$$

Energy/Temperature Considerations - Velocity-Temperature Relationships (Mean)



M = 2.5, Tw/Tr = 1.0 and Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Mean velocity-temperature relationships.

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Energy/Temperature Considerations - Velocity-Temperature Relationships (Fluctuations)

- Morkovin (1962) identified a set of velocity-temperature fluctuation correlations referred to as the strong Reynolds analogy (SRA)
  - SRA is a precursor to the GRA
- Arises from a "strong" analogy between total enthalpy and velocity: $H' = U_w u_1'$
- A common form is a root-mean-square (RMS) form
- Modifications to the SRA exist to improve result for diabatic walls

$$\frac{\dfrac{\sqrt{\overline{T'^2}}}{\overline{T}}}{(\gamma - 1)M^2 \dfrac{\sqrt{\overline{u_1'^2}}}{\overline{u_1}}} = 1$$

Basic SRA

$$\frac{\dfrac{\sqrt{\overline{T'^2}}}{\overline{T}}}{(\gamma - 1)M^2 \dfrac{\sqrt{\overline{u_1'^2}}}{\overline{u_1}}} = \frac{1}{a\left|1 - \dfrac{\partial \overline{T_t}}{\partial \overline{T}}\right|}$$

Modified SRA including total temperature gradient term

$$T_t = T + \frac{u_1^2}{2c_p}$$

Energy/Temperature Considerations - Velocity-Temperature Relationships (Fluctuations)

- Proposed parameters for modified SRA

Reference	Acronym	a
Morkovin (1962)	SRA	N/A
Gaviglio (1987)	GSRA	1
Rubesin (1990)	RSRA	1.34
Huang et al. (1995)	HSRA	Pr_t
Zhang et al. (2014)	MHRSA	$\overline{Pr_t}$

$$\overline{Pr_e} = \frac{\overline{Pr_t}}{1+\epsilon} \approx 1$$

$$\overline{Pr_t} = \frac{\overline{(\rho u_2)'u_1'}}{\overline{(\rho u_2)'T'}} \frac{\frac{\partial \overline{T}}{\partial y}}{\frac{\partial \overline{u_1}}{\partial y}} = Pr_t \frac{1 + \overline{u_2}\,\overline{\rho' u_1'}/\overline{\rho u_2' u_1'}}{1 + \overline{u_2}\,\overline{\rho' T'}/\overline{\rho u_2' T'}}$$

$$Pr_t = \frac{\overline{\rho u_2' u_1'}\,\frac{\partial \overline{T}}{\partial x_2}}{\overline{\rho u_2' T'}\,\frac{\partial \overline{u_1}}{\partial x_2}}$$

$$\epsilon = \frac{\overline{(\rho u_2)'\phi'}}{\overline{(\rho u_2)'T'}}$$

$$\phi' = \frac{-Pr\overline{q_w}}{c_p\overline{\tau_w}} u_1' - T' - \frac{r_g \overline{u_1}}{c_p} u_1'$$

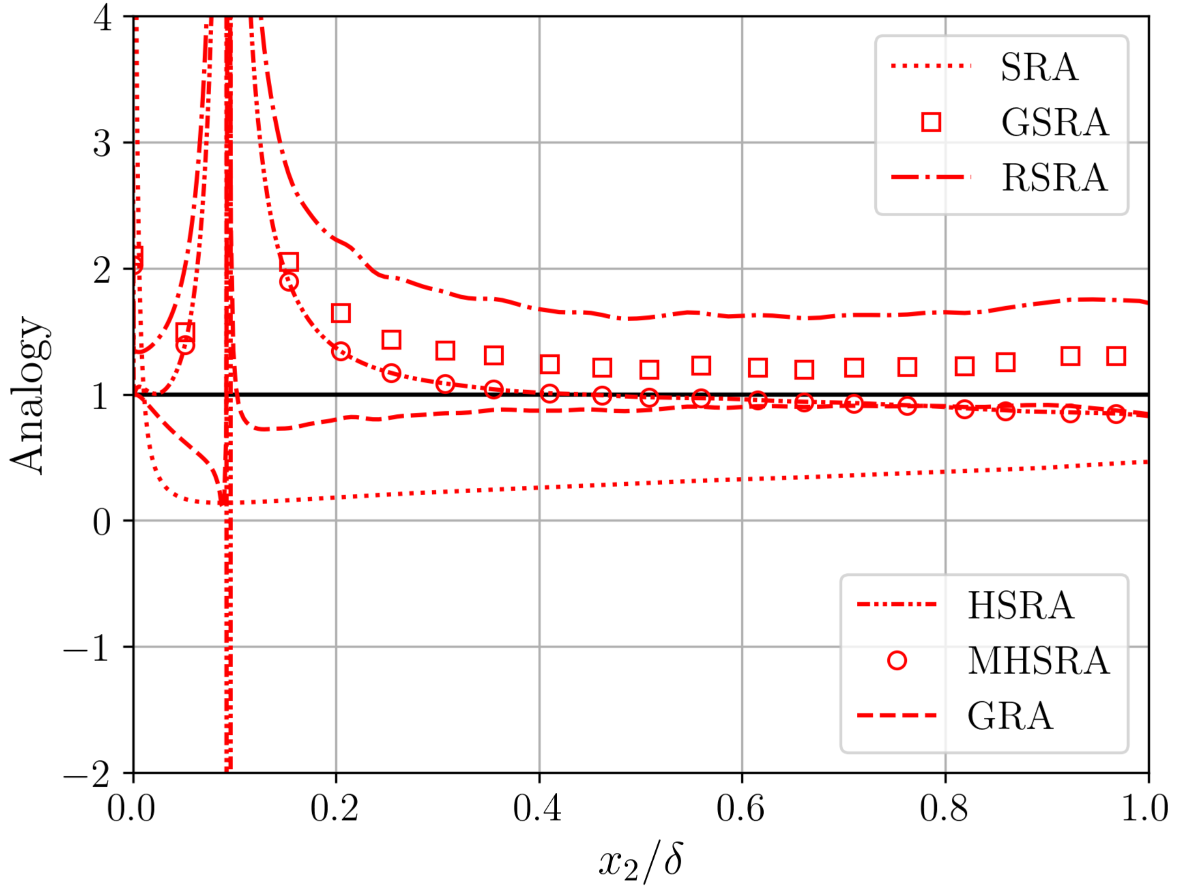
$$r_g = \frac{2c_p}{\overline{u_1}^2}\left(\overline{T_w} - \overline{T} - \frac{Pr\overline{q_w}}{c_p\overline{\tau_w}}\overline{u_1}\right)$$

- GRA in RMS form:

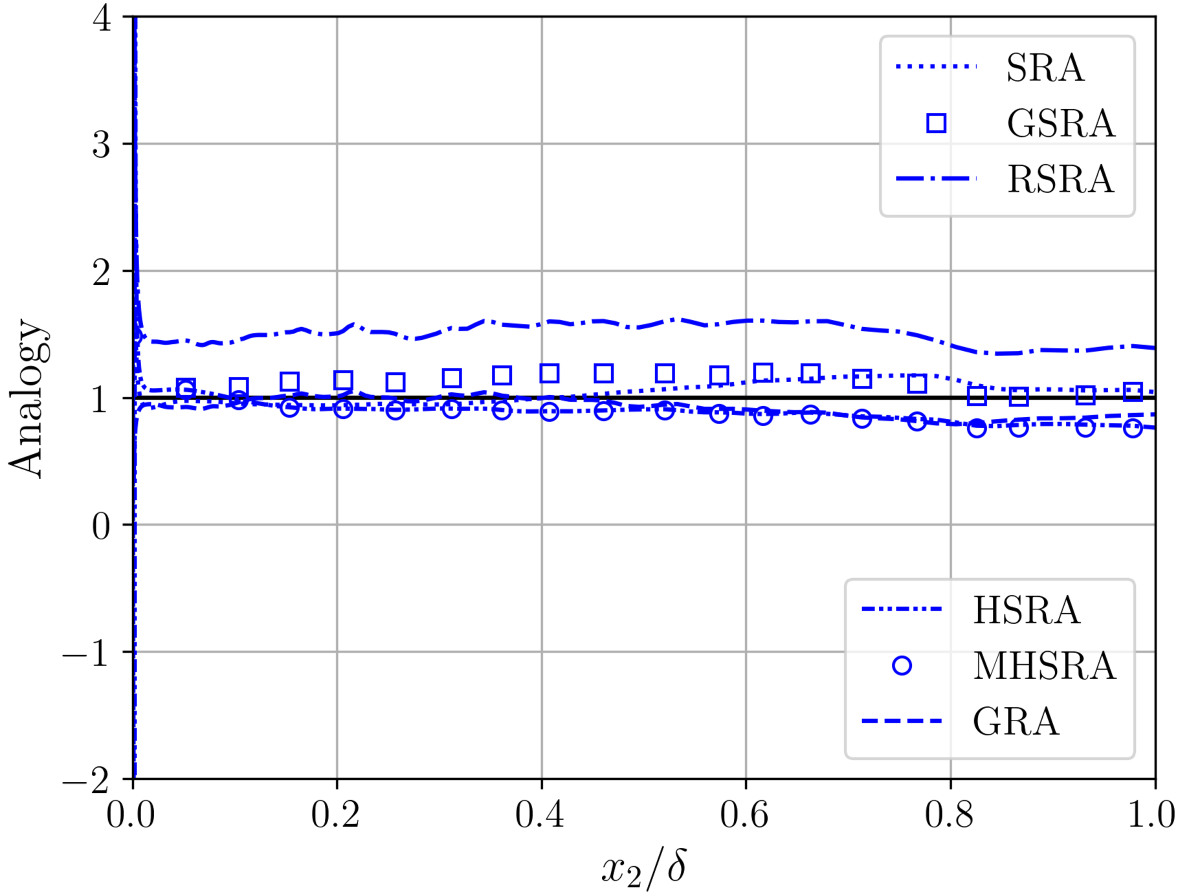
$$\sqrt{\overline{(T' + \phi')^2}} = \frac{1}{\overline{Pr_e}} \left|\frac{\partial \overline{T}}{\partial \overline{u_1}}\right| \sqrt{\overline{u_1'^2}}$$

$$\sqrt{\overline{(T' + \phi')^2}} = \left|\frac{\partial \overline{T}}{\partial \overline{u_1}}\right| \sqrt{\overline{u_1'^2}} \quad (\text{if } \overline{Pr_e} = 1)$$

Energy/Temperature Considerations - Velocity-Temperature Relationships (Fluctuations)



M = 2.5, Tw/Tr = 0.4, zero-pressure gradient flat plate turbulent boundary layer DNS. Various Reynolds analogies for RMS of fluctuations.



M = 2.5, Tw/Tr = 1.0, zero-pressure gradient flat plate turbulent boundary layer DNS. Various Reynolds analogies for RMS of fluctuations.

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