1 Introduction

The following document is intended to overview turbulence in compressible flows. Specifically continuum fluids with the perfect perfect gas assumption that obey the Navier-Stokes equations. The document is organized as follows: In Section 2 the general formulation and derivation of density-weighted (Favre) averaged quantities is presented. A brief discussion of Reynolds-averaged (non-density weighted average) is provided in Section 3 for the conservation of mass equation. Section 4 shows the Favre-averaged governing equations for conservation of mass, momentum, energy, and an additional turbulent kinetic energy equation. Within Section 4 terms to model that arise from the decomposition and averaging of the governing equations are presented and classical modeling choices discussed; however, complete turbulence models are not included presently. Following the mathematical descriptions and turbulence modeling, the subsequent sections address the analysis of wall-bounded turbulence. Section 5 includes both velocity and temperature compressibility transformations for the law of the wall. Finally, Section 6 highlights the relationships between velocity and temperature for both the mean flow and turbulent fluctuations.

Standard nomenclature is used throughout this document. Unless otherwise noted, index (summation) notation is utilized where x_1, x_2, and x_3 correspond the streamwise, wall-normal, and spanwise directions, receptively. Likewise u_1, u_2, and u_3 correspond the streamwise, wall-normal, and spanwise velocities.

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2 Favre Averaging Formulation

The Favre-averaged quantities make use of the standard decomposition of an instantaneous value into a mean $\overline{(\cdot)}$ or $\widetilde{(\cdot)}$, and a fluctuation $(\cdot)'$ or $(\cdot)''$. To demonstrate the formulation, the density (ρ) weighted average is shown for the arbitrary parameters ϕ or ψ.

Decompositions:

$$\phi = \overline{\phi} + \phi' \qquad \text{(Reynolds decomposition)} \qquad \overline{\phi'} = 0 \tag{1}$$

$$\phi = \widetilde{\phi} + \phi'' \qquad \text{(Favre decomposition)} \tag{2}$$

When Favre averaging, density, pressure, and transport properties use Reynolds decomposition, all other variables like velocity, temperature, enthalpy, energy, etc. use the Favre decomposition.

Averaging Rules:

$$\overline{\phi + \psi} = \overline{\phi} + \overline{\psi} \tag{3}$$

$$\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \overline{\phi}}{\partial x} \tag{4}$$

$$\overline{\overline{\phi}\psi} = \overline{\phi}\,\overline{\psi} \tag{5}$$

$$\overline{\widetilde{\phi}} = \widetilde{\phi} \tag{6}$$

$$\overline{\rho\widetilde{\phi}} = \overline{\rho}\widetilde{\phi} \tag{7}$$

$$\widetilde{\rho\phi} = \overline{\rho}\widetilde{\phi} \tag{8}$$

Favre Average, $\widetilde{\phi}$:

$$\widetilde{\phi} = \frac{\overline{\rho\phi}}{\overline{\rho}} \tag{9}$$

$$\overline{\rho}\widetilde{\phi} = \overline{\left(\overline{\rho} + \rho'\right)\left(\overline{\phi} + \phi'\right)} = \overline{\overline{\rho}\;\overline{\phi}} + \overline{\overline{\rho}\phi'} + \overline{\rho'\overline{\phi}} + \overline{\rho'\phi'} \tag{10}$$

$$\overline{\rho}\widetilde{\phi} = \overline{\rho}\;\overline{\phi} + \overline{\rho}\cancelto{0}{\overline{\phi'}} + \overline{\phi}\cancelto{0}{\overline{\rho'}} + \overline{\rho'\phi'} = \overline{\rho}\;\overline{\phi} + \overline{\rho'\phi'} \tag{11}$$

$$\widetilde{\phi} = \overline{\phi} + \frac{\overline{\rho'\phi'}}{\overline{\rho}} \tag{12}$$

$\overline{\rho\phi''} = 0$:

$$\rho\phi = \rho(\widetilde{\phi} + \phi'') \tag{13}$$

$$\overline{\rho\phi} = \overline{\rho(\widetilde{\phi} + \phi'')} \tag{14}$$

$$\overline{\rho\phi} = \overline{\rho}\widetilde{\phi} + \overline{\rho\phi''} \tag{15}$$

$$\widetilde{\phi} = \frac{\overline{\rho\phi}}{\overline{\rho}} = \frac{\cancel{\overline{\rho}}\widetilde{\phi}}{\cancel{\overline{\rho}}} + \frac{\overline{\rho\phi''}}{\overline{\rho}} \tag{16}$$

$$\widetilde{\phi} = \widetilde{\phi} + \frac{\overline{\rho\phi''}}{\overline{\rho}} \tag{17}$$

$$\frac{\overline{\rho\phi''}}{\overline{\rho}} = 0 \qquad (\overline{\rho} \geq 0) \tag{18}$$

$$\overline{\rho\phi''} = 0 \tag{19}$$

$\overline{\phi''} \neq 0$:

$$\overline{\phi} = \overline{\widetilde{\phi} + \phi''} \tag{20}$$

$$\overline{\phi} = \overline{\widetilde{\phi}} + \overline{\phi''} \tag{21}$$

$$\overline{\phi''} = \overline{\phi} - \widetilde{\phi} \tag{22}$$

$$\overline{\phi''} = \overline{\phi} - \frac{\overline{\rho\phi}}{\overline{\rho}} \tag{23}$$

$$\overline{\phi''} = \frac{\overline{\rho}\;\overline{\phi}}{\overline{\rho}} - \frac{\overline{\rho\phi}}{\overline{\rho}} \tag{24}$$

$$\overline{\phi''} = \frac{\overline{\overline{\rho}\;\overline{\phi}}}{\overline{\rho}} - \frac{\overline{(\overline{\rho} + \rho')(\overline{\phi} + \phi')}}{\overline{\rho}} \tag{25}$$

$$\overline{\phi''} = \frac{\cancel{\overline{\overline{\rho}\;\overline{\phi}}} - \cancel{\overline{\overline{\rho}\;\overline{\phi}}} - \cancelto{0}{\overline{\overline{\rho}\phi'}} - \cancelto{0}{\overline{\rho'\overline{\phi}}} - \overline{\rho'\phi'}}{\overline{\rho}} \tag{26}$$

$$\overline{\phi''} = \frac{-\overline{\rho'\phi'}}{\overline{\rho}} \neq 0 \tag{27}$$

$\overline{\rho\phi\psi} = \overline{\rho}\widetilde{\phi}\widetilde{\psi} + \overline{\rho\phi''\psi''}$:

$$\overline{\rho\phi\psi} = \overline{\rho(\widetilde{\phi} + \phi'')(\widetilde{\psi} + \psi'')} \tag{28}$$
$$\overline{\rho\phi\psi} = \overline{\rho\widetilde{\phi}\widetilde{\psi} + \rho\phi''\widetilde{\psi} + \rho\widetilde{\phi}\psi'' + \rho\phi''\psi''} \tag{29}$$
$$\overline{\rho\phi\psi} = \overline{\rho\widetilde{\phi}\widetilde{\psi}} + \overline{\rho\phi''\widetilde{\psi}} + \overline{\rho\widetilde{\phi}\psi''} + \overline{\rho\phi''\psi''} \tag{30}$$
$$\overline{\rho\phi\psi} = \overline{\rho}\widetilde{\phi}\widetilde{\psi} + \cancelto{0}{\overline{\rho\phi''}}\widetilde{\psi} + \cancelto{0}{\overline{\rho\psi''}}\widetilde{\phi} + \overline{\rho\phi''\psi''} \tag{31}$$
$$\overline{\rho\phi\psi} = \overline{\rho}\widetilde{\phi}\widetilde{\psi} + \overline{\rho\phi''\psi''} \tag{32}$$

3 Reynolds Averaged Governing Equations

To show the benefit of Favre averaging, the Reynolds-averaged conservation of mass governing equation is shown next. The primary outcome of this demonstration is to show that standard averaging leads to an extra density-velocity fluctuation correlation that would have been absorbed into the mean if a density-weighted average had been used. In this section ρ is density, u_j is the velocity vector, t is time, and x_j is the spatial coordinate.

3.1 Continuity

Instantaneous:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \tag{33}$$

Decompose and Average:

$$\overline{\frac{\partial(\overline{\rho} + \rho')}{\partial t}} + \overline{\frac{\partial(\overline{\rho} + \rho')(\overline{u_j} + u_j')}{\partial x_j}} = 0 \tag{34}$$
$$\frac{\partial\overline{\overline{\rho}}}{\partial t} + \frac{\partial\overline{\rho'}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\overline{\rho}\,\overline{u_j}} + \overline{\overline{\rho}u_j'} + \overline{\rho'\overline{u_j}} + \overline{\rho' u_j'}\right) = 0 \tag{35}$$
$$\frac{\partial\overline{\rho}}{\partial t} + \frac{\partial\cancelto{0}{\overline{\rho'}}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\rho}\,\overline{u_j} + \overline{\rho}\cancelto{0}{\overline{u_j'}} + \cancelto{0}{\overline{\rho'}}\overline{u_j} + \overline{\rho' u_j'}\right) = 0 \tag{36}$$
$$\frac{\partial\overline{\rho}}{\partial t} + \frac{\partial\overline{\rho}\,\overline{u_j}}{\partial x_j} + \frac{\partial\overline{\rho' u_j'}}{\partial x_j} = 0 \tag{37}$$

4 Favre Averaged Governing Equations

This section shows the Favre-averaged (density-weighted) conservation equations for mass, momentum, energy, and turbulent kinetic energy (TKE). In this section ρ is density, u_i is the velocity vector, P is static pressure, T is static temperature, μ is dynamic viscosity, t is time, and x_i is the spatial coordinate. Other parameters will be defined as they arise.

With a focus on turbulence modeling research, terms necessary to model to close the system of equations are identified, and classical modeling choices are presented.

4.1 Continuity

Instantaneous:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} = 0 \tag{38}$$

Decompose and Average:

$$\overline{\frac{\partial(\overline{\rho} + \rho')}{\partial t}} + \overline{\frac{\partial(\overline{\rho} + \rho')(\widetilde{u_j} + u_j'')}{\partial x_j}} = 0 \tag{39}$$

$$\frac{\partial \overline{\overline{\rho}}}{\partial t} + \frac{\partial \overline{\rho'}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\overline{\rho}\widetilde{u_j}} + \overline{\overline{\rho} u_j''} + \overline{\rho' \widetilde{u_j}} + \overline{\rho' u_j''}\right) = 0 \tag{40}$$

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \cancelto{0}{\overline{\rho'}}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\rho}\widetilde{u_j} + \cancelto{0}{\overline{\rho'}}\widetilde{u_j} + \overline{(\overline{\rho} + \rho')u_j''}\right) = 0 \tag{41}$$

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\rho}\widetilde{u_j} + \cancelto{0}{\overline{\rho u_j''}}\right) = 0 \tag{42}$$

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}\widetilde{u_j}}{\partial x_j} = 0 \tag{43}$$

4.2 Momentum

Instantaneous:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial t_{ij}}{\partial x_j} \tag{44}$$

Viscous Stress Tensor:

$$t_{ij} = 2\mu S_{ij} + \lambda \frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{45}$$

$$t_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{46}$$

$$\lambda = -\frac{2}{3}\mu \quad \text{(Stokes' hypothesis)} \tag{47}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{(Strain rate tensor)} \tag{48}$$

Decompositions:

$$\rho = \overline{\rho} + \rho' \tag{49}$$

$$P = \overline{P} + P' \tag{50}$$

$$\mu = \overline{\mu} + \mu' \tag{51}$$

$$\lambda = \overline{\lambda} + \lambda' \tag{52}$$

$$u_i = \widetilde{u_i} + u_i'' \tag{53}$$

Decompose and Average:

$$\overline{\frac{\partial(\overline{\rho}+\rho')(\widetilde{u_i}+u_i'')}{\partial t}} + \overline{\frac{\partial(\overline{\rho}+\rho')(\widetilde{u_i}+u_i'')(\widetilde{u_j}+u_j'')}{\partial x_j}} = -\overline{\frac{\partial(\overline{P}+P')}{\partial x_i}} + \overline{\frac{\partial t_{ij}}{\partial x_j}} \tag{54}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\overline{\overline{\rho}\widetilde{u_i}} + \overline{\overline{\rho}u_i''} + \overline{\rho'\widetilde{u_i}} + \overline{\rho' u_i''}\right] \\
&\quad + \frac{\partial}{\partial x_j}\left[\overline{\overline{\rho}\widetilde{u_i}\widetilde{u_j}} + \overline{\overline{\rho}u_i''\widetilde{u_j}} + \overline{\rho'\widetilde{u_i}\widetilde{u_j}} + \overline{\rho' u_i''\widetilde{u_j}} + \overline{\overline{\rho}\widetilde{u_i}u_j''} + \overline{\overline{\rho}u_i''u_j''} + \overline{\rho'\widetilde{u_i}u_j''} + \overline{\rho' u_i'' u_j''}\right] \\
&\quad = -\frac{\partial}{\partial x_i}\left[\overline{\overline{P}} + \cancelto{0}{\overline{P'}}\right] + \frac{\partial \overline{t_{ij}}}{\partial x_j}
\end{aligned} \tag{55}$$

$$\frac{\partial}{\partial t}\left[\overline{\rho}\widetilde{u_i} + \cancelto{0}{\overline{\rho u_i''}} + \cancelto{0}{\overline{\rho'}}\widetilde{u_i}\right] + \frac{\partial}{\partial x_j}\left[\overline{\rho}\widetilde{u_i}\widetilde{u_j} + \cancelto{0}{\overline{\rho u_i''}}\widetilde{u_j} + \cancelto{0}{\overline{\rho'}}\widetilde{u_i}\widetilde{u_j} + \cancelto{0}{\overline{\rho u_j''}}\widetilde{u_i} + \overline{\rho u_i'' u_j''}\right] = -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial \overline{t_{ij}}}{\partial x_j} \tag{56}$$

$$\frac{\partial}{\partial t}\left[\overline{\rho}\widetilde{u_i}\right] + \frac{\partial}{\partial x_j}\left[\overline{\rho}\widetilde{u_i}\widetilde{u_j} + \overline{\rho u_i'' u_j''}\right] = -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial \overline{t_{ij}}}{\partial x_j} \tag{57}$$

$$\frac{\partial(\overline{\rho}\widetilde{u_i})}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{u_i}\widetilde{u_j})}{\partial x_j} = -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial \overline{t_{ij}}}{\partial x_j} - \frac{\partial(\overline{\rho u_i'' u_j''})}{\partial x_j} \tag{58}$$

Mean Viscous Stress Tensor, $\overline{\mathbf{t_{ij}}}$:

$$\overline{t_{ij}} = \overline{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}} \tag{59}$$

$$\overline{t_{ij}} = \overline{(\overline{\mu}+\mu')\left(\frac{\partial(\widetilde{u_i}+u_i'')}{\partial x_j} + \frac{\partial(\widetilde{u_j}+u_j'')}{\partial x_i}\right) + (\overline{\lambda}+\lambda')\frac{\partial(\widetilde{u_k}+u_k'')}{\partial x_k}\delta_{ij}} \tag{60}$$

$$\begin{aligned}
\overline{t_{ij}} = &\left[\overline{\overline{\mu}\frac{\partial\widetilde{u_i}}{\partial x_j}} + \overline{\overline{\mu}\frac{\partial u_i''}{\partial x_j}} + \overline{\mu'\frac{\partial\widetilde{u_i}}{\partial x_j}} + \overline{\mu'\frac{\partial u_i''}{\partial x_j}} + \overline{\overline{\mu}\frac{\partial\widetilde{u_j}}{\partial x_i}} + \overline{\overline{\mu}\frac{\partial u_j''}{\partial x_i}} + \overline{\mu'\frac{\partial\widetilde{u_j}}{\partial x_i}} + \overline{\mu'\frac{\partial u_j''}{\partial x_i}}\right] \\
&+ \delta_{ij}\left[\overline{\overline{\lambda}\frac{\partial\widetilde{u_k}}{\partial x_k}} + \overline{\overline{\lambda}\frac{\partial u_k''}{\partial x_k}} + \overline{\lambda'\frac{\partial\widetilde{u_k}}{\partial x_k}} + \overline{\lambda'\frac{\partial u_k''}{\partial x_k}}\right]
\end{aligned} \tag{61}$$

$$\overline{t_{ij}} = \left[\overline{\mu}\frac{\partial\widetilde{u_i}}{\partial x_j} + \overline{\mu\frac{\partial u_i''}{\partial x_j}} + \cancelto{0}{\overline{\mu'}}\frac{\partial\widetilde{u_i}}{\partial x_j} + \overline{\mu}\frac{\partial\widetilde{u_j}}{\partial x_i} + \overline{\mu\frac{\partial u_j''}{\partial x_i}} + \cancelto{0}{\overline{\mu'}}\frac{\partial\widetilde{u_j}}{\partial x_i}\right] + \delta_{ij}\left[\overline{\lambda}\frac{\partial\widetilde{u_k}}{\partial x_k} + \overline{\lambda\frac{\partial u_k''}{\partial x_k}} + \cancelto{0}{\overline{\lambda'}}\frac{\partial\widetilde{u_k}}{\partial x_k}\right] \tag{62}$$

$$\overline{t_{ij}} = \underbrace{\overline{\mu}\left(\frac{\partial\widetilde{u_i}}{\partial x_j} + \frac{\partial\widetilde{u_j}}{\partial x_i}\right) + \overline{\lambda}\frac{\partial\widetilde{u_k}}{\partial x_k}\delta_{ij}}_{\widetilde{t_{ij}}} + \underbrace{\overline{\mu\left(\frac{\partial u_i''}{\partial x_j} + \frac{\partial u_j''}{\partial x_i}\right) + \lambda\frac{\partial u_k''}{\partial x_k}\delta_{ij}}}_{t_{ij}''} \tag{63}$$

$$\overline{t_{ij}} = \widetilde{t_{ij}} \qquad \left(\text{assuming } |\widetilde{t_{ij}}| \gg |t_{ij}''|\right) \tag{64}$$

Favre-averaged Reynolds stress tensor, $\tau_{\mathbf{ij}}$:

$$\tau_{ij} = \frac{-\overline{\rho u_i'' u_j''}}{\overline{\rho}} \tag{65}$$

4.3 Energy

Instantaneous:

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{u_i u_i}{2}\right)\right] + \frac{\partial}{\partial x_j}\left[\rho u_j\left(h + \frac{u_i u_i}{2}\right)\right] = \frac{\partial(u_i t_{ij})}{\partial x_j} - \frac{\partial q_j}{\partial x_j} \tag{66}$$

Specific internal energy (e) and specific enthalpy (h):

$$h = e + \frac{P}{\rho} \tag{67}$$

$$e = c_v T \quad \text{(calorically perfect, } c_v \text{ constant)} \tag{68}$$

$$h = c_p T \quad \text{(calorically perfect, } c_p \text{ constant)} \tag{69}$$

Heat Flux Vector from Fourier's Law $\mathbf{q_j}$ and thermal conductivity κ:

$$q_j = -\kappa \frac{\partial T}{\partial x_j} = -\frac{\mu}{Pr_L} \frac{\partial h}{\partial x_j} \tag{70}$$

$$\kappa = \frac{\mu c_p}{Pr_L} \quad (Pr_L \text{ is laminar Prandtl number}) \tag{71}$$

$$q_{Lj} = -\overline{\kappa \frac{\partial T}{\partial x_2}} \approx -\overline{\kappa} \frac{\partial \overline{T}}{\partial x_j} \quad \text{(molecular heat flux)} \tag{72}$$

$$\frac{\partial q_{Lj}}{\partial x_j} = -\frac{\partial}{\partial x_j} \left[\overline{\kappa \frac{\partial T}{\partial x_2}} \right] \quad \text{(molecular diffusion/transport of heat)} \tag{73}$$

Viscous stress tensor:

$$t_{ij} = 2\mu S_{ij} + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} \tag{74}$$

$$t_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} \tag{75}$$

$$\lambda = -\frac{2}{3}\mu \quad \text{(Stokes' hypothesis)} \tag{76}$$

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{(Strain rate tensor)} \tag{77}$$

Decompositions:

$$\rho = \overline{\rho} + \rho' \tag{78}$$

$$P = \overline{P} + P' \tag{79}$$

$$\mu = \overline{\mu} + \mu' \tag{80}$$

$$\lambda = \overline{\lambda} + \lambda' \tag{81}$$

$$\kappa = \overline{\kappa} + \kappa' \tag{82}$$

$$u_i = \widetilde{u_i} + u_i'' \tag{83}$$

$$e = \widetilde{e} + e'' \tag{84}$$

$$h = \widetilde{h} + h'' \tag{85}$$

$$T = \widetilde{T} + T'' \tag{86}$$

$$q_j = q_{Lj} + q_j' \quad (q_{Lj} \equiv \overline{q_j}) \tag{87}$$

Decompose and Average:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\overline{\left[(\overline{\rho}+\rho')\left(\widetilde{e}+e''+\frac{1}{2}(\widetilde{u}_i+u_i'')(\widetilde{u}_i+u_i'')\right)\right]}+\frac{\partial}{\partial x_j}\overline{\left[(\overline{\rho}+\rho')(\widetilde{u}_j+u_j'')\left((\widetilde{h}+h'')+\frac{1}{2}(\widetilde{u}_i+u_i'')(\widetilde{u}_i+u_i'')\right)\right]} \\
&\quad=\frac{\partial}{\partial x_j}\overline{[(\widetilde{u}_i+u_i'')t_{ij}]}-\frac{\partial}{\partial x_j}\overline{[q_{Lj}+q_j']}
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\overline{\overline{\rho}\widetilde{e}}+\overline{\overline{\rho}e''}+\overline{\rho'\widetilde{e}}+\overline{\rho'e''}+\frac{1}{2}\left(\overline{\overline{\rho}\widetilde{u}_i\widetilde{u}_i}+2\overline{\overline{\rho}\widetilde{u}_iu_i''}+\overline{\overline{\rho}u_i''u_i''}+\overline{\rho'\widetilde{u}_i\widetilde{u}_i}+2\overline{\rho'\widetilde{u}_iu_i''}+\overline{\rho'u_i''u_i''}\right)\right] \\
&\quad+\frac{\partial}{\partial x_j}\left[\overline{\overline{\rho}\widetilde{u}_j\widetilde{h}}+\overline{\overline{\rho}\widetilde{u}_jh''}+\frac{1}{2}\left(\overline{\overline{\rho}\widetilde{u}_j\widetilde{u}_i\widetilde{u}_i}+2\overline{\overline{\rho}\widetilde{u}_j\widetilde{u}_iu_i''}+\overline{\overline{\rho}\widetilde{u}_ju_i''u_i''}\right)+\right. \\
&\qquad\qquad\overline{\overline{\rho}u_j''\widetilde{h}}+\overline{\overline{\rho}u_j''h''}+\frac{1}{2}\left(\overline{\overline{\rho}u_j''\widetilde{u}_i\widetilde{u}_i}+2\overline{\overline{\rho}u_j''\widetilde{u}_iu_i''}+\overline{\overline{\rho}u_j''u_i''u_i''}\right)+ \\
&\qquad\qquad\overline{\rho'\widetilde{u}_j\widetilde{h}}+\overline{\rho'\widetilde{u}_jh''}+\frac{1}{2}\left(\overline{\rho'\widetilde{u}_j\widetilde{u}_i\widetilde{u}_i}+2\overline{\rho'\widetilde{u}_j\widetilde{u}_iu_i''}+\overline{\rho\widetilde{u}_ju_i''u_i''}\right)+ \\
&\qquad\qquad\left.\overline{\rho'u_j''\widetilde{h}}+\overline{\rho'u_j''h''}+\frac{1}{2}\left(\overline{\rho'u_j''\widetilde{u}_i\widetilde{u}_i}+2\overline{\rho'u_j''\widetilde{u}_iu_i''}+\overline{\rho'u_j''u_i''u_i''}\right)\right] \\
&\quad=\frac{\partial}{\partial x_j}\left[\overline{\widetilde{u}_it_{ij}}+\overline{u_i''t_{ij}}\right]-\frac{\partial}{\partial x_j}\left[\overline{q_{Lj}}+\cancelto{0}{\overline{q_j'}}\right]
\end{aligned}
\tag{89}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\overline{\rho}\widetilde{e}+\cancelto{0}{\overline{\rho e''}}+\cancelto{0}{\overline{\rho'}}\widetilde{e}+\frac{1}{2}\left(\overline{\rho}\widetilde{u}_i\widetilde{u}_i+2\cancelto{0}{\overline{\rho u_i''}}\widetilde{u}_i+\overline{\rho u_i''u_i''}+\cancelto{0}{\overline{\rho'}}\widetilde{u}_i\widetilde{u}_i\right)\right] \\
&\quad+\frac{\partial}{\partial x_j}\left[\overline{\rho}\widetilde{u}_j\widetilde{h}+\cancelto{0}{\overline{\rho h''}}\widetilde{u}_j+\cancelto{0}{\overline{\rho u_j''}}\widetilde{h}+\cancelto{0}{\overline{\rho'}}\widetilde{u}_j\widetilde{h}_j+\overline{\rho u_j''h''}+\right. \\
&\qquad\left.\frac{1}{2}\left(\overline{\rho}\widetilde{u}_j\widetilde{u}_i\widetilde{u}_i+2\cancelto{0}{\overline{\rho u_i''}}\widetilde{u}_j\widetilde{u}_i+\overline{\rho u_i''u_i''}\widetilde{u}_j+\cancelto{0}{\overline{\rho u_j''}}\widetilde{u}_i\widetilde{u}_i+\overline{\rho u_j''u_i''u_i''}+2\overline{\rho u_j''u_i''}\widetilde{u}_i+\cancelto{0}{\overline{\rho'}}\widetilde{u}_j\widetilde{u}_i\widetilde{u}_i\right)\right]+ \\
&\quad=\frac{\partial}{\partial x_j}\left[\widetilde{u}_i\overline{t_{ij}}+\overline{u_i''t_{ij}}\right]-\frac{\partial}{\partial x_j}\left[q_{Lj}\right]
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\overline{\rho}\widetilde{e}+\frac{1}{2}\left(\overline{\rho}\widetilde{u}_i\widetilde{u}_i+\overline{\rho u_i''u_i''}\right)\right]+\frac{\partial}{\partial x_j}\left[\overline{\rho}\widetilde{u}_j\widetilde{h}+\overline{\rho u_j''h''}+\frac{1}{2}\left(\overline{\rho}\widetilde{u}_j\widetilde{u}_i\widetilde{u}_i+\overline{\rho u_i''u_i''}\widetilde{u}_j+\overline{\rho u_j''u_i''u_i''}+2\overline{\rho u_j''u_i''}\widetilde{u}_i\right)\right] \\
&\quad=\frac{\partial}{\partial x_j}\left[\widetilde{u}_i\overline{t_{ij}}+\overline{u_i''t_{ij}}\right]-\frac{\partial}{\partial x_j}\left[q_{Lj}\right]
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\overline{\rho}\left(\widetilde{e}+\frac{\widetilde{u}_i\widetilde{u}_i}{2}\right)+\frac{\overline{\rho u_i''u_i''}}{2}\right]+\frac{\partial}{\partial x_j}\left[\overline{\rho}\widetilde{u}_j\left(\widetilde{h}+\frac{\widetilde{u}_i\widetilde{u}_i}{2}\right)+\widetilde{u}_j\frac{\overline{\rho u_i''u_i''}}{2}\right] \\
&\quad=\frac{\partial}{\partial x_j}\left[\widetilde{u}_i\left(\overline{t_{ij}}-\overline{\rho u_i''u_j''}\right)\right]-\frac{\partial}{\partial x_j}\left[q_{Lj}+\overline{\rho u_j''h''}-\overline{u_i''t_{ij}}+\frac{\overline{\rho u_j''u_i''u_i''}}{2}\right]
\end{aligned}
\tag{92}
$$

Turbulent kinetic energy per unit volume, k:

$$
k=\frac{\overline{\rho u_i''u_i''}}{2\overline{\rho}}
\tag{93}
$$

Favre-averaged Reynolds stress tensor, τ_{ij}:

$$
\tau_{ij}=\frac{-\overline{\rho u_i''u_j''}}{\overline{\rho}}
\tag{94}
$$

Turbulent transport of heat, $\frac{\partial \mathbf{q_{Tj}}}{\partial \mathbf{x_j}}$:

$$q_{Tj} = \overline{\rho u_j'' h''} \quad \text{(turbulent heat flux)} \tag{95}$$

4.4 Ideal Gas Equation of State

Instantaneous:

$$P = \rho R T \qquad (R \text{ is perfect gas constant}) \tag{96}$$

Decompositions:

$$\rho = \overline{\rho} + \rho' \tag{97}$$

$$P = \overline{P} + P' \tag{98}$$

$$T = \widetilde{T} + T'' \tag{99}$$

Decompose and Average:

$$\overline{(\overline{P} + P')} = \overline{(\overline{\rho} + \rho')R(\widetilde{T} + T'')} \tag{100}$$

$$\overline{\overline{P}} + \cancelto{0}{\overline{P'}} = \overline{(\overline{\rho}\widetilde{T} + \overline{\rho}T'' + \rho'\widetilde{T} + \rho'T'')R} \tag{101}$$

$$\overline{P} = (\overline{\overline{\rho}\widetilde{T}} + \cancelto{0}{\overline{\rho T''}} + \overline{\rho'\widetilde{T}})R \tag{102}$$

$$\overline{P} = (\overline{\rho}\widetilde{T} + \cancelto{0}{\overline{\rho'}}\widetilde{T})R \tag{103}$$

$$\overline{P} = \overline{\rho} R \widetilde{T} \tag{104}$$

4.5 Turbulent Kinetic Energy

The Favre-averaged governing equation for the turbulent kinetic energy (TKE) is found by multiplying the Favre fluctuation for velocity to the momentum equation in primitive variable formulation, and then decomposing and averaging as usual.

Instantaneous:

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial t_{ij}}{\partial x_j} \tag{105}$$

$$\rho u_i'' \frac{\partial u_i}{\partial t} + \rho u_i'' u_j \frac{\partial u_i}{\partial x_j} = -u_i'' \frac{\partial P}{\partial x_i} + u_i'' \frac{\partial t_{ij}}{\partial x_j} \tag{106}$$

Viscous Stress Tensor:

$$t_{ij} = 2\mu S_{ij} + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} \tag{107}$$

$$t_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} \tag{108}$$

$$\lambda = -\frac{2}{3}\mu \qquad \text{(Stokes' hypothesis)} \tag{109}$$

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad \text{(Strain rate tensor)} \tag{110}$$

Decompositions:

$$\rho = \overline{\rho} + \rho' \tag{111}$$

$$P = \overline{P} + P' \tag{112}$$

$$\mu = \overline{\mu} + \mu' \tag{113}$$

$$\lambda = \overline{\lambda} + \lambda' \tag{114}$$

$$u_i = \widetilde{u}_i + u_i'' \tag{115}$$

Decompose and Average:

$$\overline{\rho u_i'' \frac{\partial(\widetilde{u}_i + u_i'')}{\partial t}} + \overline{\rho u_i''(\widetilde{u}_j + u_j'')\frac{\partial(\widetilde{u}_i + u_i'')}{\partial x_j}} = -\overline{u_i'' \frac{\partial(\overline{P} + P')}{\partial x_i}} + \overline{u_i'' \frac{\partial t_{ij}}{\partial x_j}} \tag{116}$$

$$\begin{aligned}&\overline{\rho u_i'' \frac{\partial \widetilde{u}_i}{\partial t}} + \overline{\rho u_i'' \frac{\partial u_i''}{\partial t}} + \overline{\rho u_i'' \widetilde{u}_j \frac{\partial \widetilde{u}_i}{\partial x_j}} + \overline{\rho u_i'' \widetilde{u}_j \frac{\partial u_i''}{\partial x_j}} + \overline{\rho u_i'' u_j'' \frac{\partial \widetilde{u}_i}{\partial x_j}} + \overline{\rho u_i'' u_j'' \frac{\partial u_i''}{\partial x_j}} \\ &\quad = -\overline{u_i'' \frac{\partial \overline{P}}{\partial x_i}} - \overline{u_i'' \frac{\partial P'}{\partial x_i}} + \frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} - \overline{t_{ij} \frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{117}$$

$$\begin{aligned}&\cancelto{0}{\overline{\rho u_i''}} \frac{\partial \widetilde{u}_i}{\partial t} + \overline{\rho \frac{1}{2} \frac{\partial u_i'' u_i''}{\partial t}} + \cancelto{0}{\overline{\rho u_i''}} \widetilde{u}_j \frac{\partial \widetilde{u}_i}{\partial x_j} + \overline{\rho u_i'' u_j \frac{\partial u_i''}{\partial x_j}} + \overline{\rho u_i'' u_j''} \frac{\partial \widetilde{u}_i}{\partial x_j} \\ &\quad = -\overline{u_i''} \frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_i'' P'}}{\partial x_i} + \overline{P' \frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} - \overline{t_{ij} \frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{118}$$

$$\begin{aligned}&\frac{\partial \overline{\rho \frac{1}{2} u_i'' u_i''}}{\partial t} - \frac{1}{2}\overline{u_i'' u_i'' \frac{\partial \rho}{\partial t}} + \overline{\rho \frac{1}{2} u_j \frac{\partial u_i'' u_i''}{\partial x_j}} - \overline{\rho} \tau_{ij} \frac{\partial \widetilde{u}_i}{\partial x_j} \\ &\quad = -\overline{u_i''} \frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_i'' P'}}{\partial x_i} + \overline{P' \frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} - \overline{t_{ij} \frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{119}$$

$$\begin{aligned}&\frac{\partial (\overline{\rho} k)}{\partial t} - \frac{1}{2}\overline{u_i'' u_i'' \frac{\partial \rho}{\partial t}} + \frac{\partial \overline{\rho \frac{1}{2} u_i'' u_i'' u_j}}{\partial x_j} - \frac{1}{2}\overline{u_i'' u_i'' \frac{\partial (\rho u_j)}{\partial x_j}} - \overline{\rho} \tau_{ij} \frac{\partial \widetilde{u}_i}{\partial x_j} \\ &\quad = -\overline{u_i''} \frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_i'' P'}}{\partial x_i} + \overline{P' \frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} - \overline{t_{ij} \frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{120}$$

$$\begin{aligned}&\frac{\partial (\overline{\rho} k)}{\partial t} - \frac{1}{2}\overline{u_i'' u_i'' \frac{\partial \rho}{\partial t}} + \frac{\partial \overline{\rho \frac{1}{2} u_i'' u_i'' (\widetilde{u}_j + u_j'')}}{\partial x_j} - \overline{\frac{1}{2} u_i'' u_i'' \frac{\partial (\rho u_j)}{\partial x_j}} - \overline{\rho} \tau_{ij} \frac{\partial \widetilde{u}_i}{\partial x_j} \\ &\quad = -\overline{u_i''} \frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_i'' P'}}{\partial x_i} + \overline{P' \frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} - \overline{t_{ij} \frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{121}$$

$$\begin{aligned}&\frac{\partial (\overline{\rho} k)}{\partial t} - \frac{1}{2}\overline{u_i'' u_i'' \frac{\partial \rho}{\partial t}} + \frac{\partial \overline{\rho \frac{1}{2} u_i'' u_i'' \widetilde{u}_j}}{\partial x_j} + \frac{\partial \overline{\rho \frac{1}{2} u_i'' u_i'' u_j''}}{\partial x_j} - \overline{\frac{1}{2} u_i'' u_i'' \frac{\partial (\rho u_j)}{\partial x_j}} - \overline{\rho} \tau_{ij} \frac{\partial \widetilde{u}_i}{\partial x_j} \\ &\quad = -\overline{u_i''} \frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_i'' P'}}{\partial x_i} + \overline{P' \frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} - \overline{t_{ij} \frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{122}$$

$$\begin{aligned}&\frac{\partial (\overline{\rho} k)}{\partial t} - \frac{1}{2}\overline{u_i'' u_i'' \frac{\partial \rho}{\partial t}} + \frac{\partial \overline{\rho \frac{1}{2} u_i'' u_i''} \widetilde{u}_j}{\partial x_j} + \frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j} - \overline{\frac{1}{2} u_i'' u_i'' \frac{\partial (\rho u_j)}{\partial x_j}} - \overline{\rho} \tau_{ij} \frac{\partial \widetilde{u}_i}{\partial x_j} \\ &\quad = -\overline{u_i''} \frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_i'' P'}}{\partial x_i} + \overline{P' \frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} - \overline{t_{ij} \frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{123}$$

$$\frac{\partial(\overline{\rho}k)}{\partial t} - \frac{1}{2}\overline{u_i''u_i''\frac{\partial \rho}{\partial t}} + \frac{\partial(\overline{\rho}k\widetilde{u}_j)}{\partial x_j} + \frac{1}{2}\overline{\frac{\partial \rho u_i''u_i''u_j''}{\partial x_j}} - \overline{\frac{1}{2}u_i''u_i''\frac{\partial(\rho u_j)}{\partial x_j}} - \overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j}$$
$$= -\overline{u_i''}\frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_i''P'}}{\partial x_i} + \overline{P'\frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i''t_{ij}}}{\partial x_j} - \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} \tag{124}$$

$$\frac{\partial(\overline{\rho}k)}{\partial t} - \frac{1}{2}\overline{u_i''u_i''\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j}\right)}^{0} + \frac{\partial(\overline{\rho}k\widetilde{u}_j)}{\partial x_j} + \frac{1}{2}\overline{\frac{\partial \rho u_i''u_i''u_j''}{\partial x_j}} - \overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j}$$
$$= -\overline{u_i''}\frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_i''P'}}{\partial x_i} + \overline{P'\frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i''t_{ij}}}{\partial x_j} - \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} \tag{125}$$

$$\frac{\partial(\overline{\rho}k)}{\partial t} + \frac{\partial(\overline{\rho}k\widetilde{u}_j)}{\partial x_j} + \frac{1}{2}\overline{\frac{\partial \rho u_i''u_i''u_j''}{\partial x_j}} - \overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} = -\overline{u_i''}\frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_j''P'}}{\partial x_j} + \overline{P'\frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i''t_{ij}}}{\partial x_j} - \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} \tag{126}$$

$$\frac{\partial(\overline{\rho}k)}{\partial t} + \frac{\partial(\overline{\rho}k\widetilde{u}_j)}{\partial x_j} = -\frac{1}{2}\frac{\partial \overline{\rho u_i''u_i''u_j''}}{\partial x_j} + \overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} - \overline{u_i''}\frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \overline{u_j''P'}}{\partial x_j} + \overline{P'\frac{\partial u_i''}{\partial x_i}} + \frac{\partial \overline{u_i''t_{ij}}}{\partial x_j} - \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} \tag{127}$$

$$\frac{\partial(\overline{\rho}k)}{\partial t} + \frac{\partial(\overline{\rho}k\widetilde{u}_j)}{\partial x_j} = \overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} - \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} + \frac{\partial}{\partial x_j}\left[\overline{u_i''t_{ij}} - \overline{u_j''P'} - \frac{1}{2}\overline{\rho u_i''u_i''u_j''}\right] - \overline{u_i''}\frac{\partial \overline{P}}{\partial x_i} + \overline{P'\frac{\partial u_i''}{\partial x_i}} \tag{128}$$

Simplifying the left-hand-side of Eq. 128 with the Favre-averaged continuity equation results in a TKE equation in primitive variable form.

$$\overline{\rho}\frac{\partial k}{\partial t} + \overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} - \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} + \frac{\partial}{\partial x_j}\left[\overline{u_i''t_{ij}} - \overline{u_j''P'} - \frac{1}{2}\overline{\rho u_i''u_i''u_j''}\right] - \overline{u_i''}\frac{\partial \overline{P}}{\partial x_i} + \overline{P'\frac{\partial u_i''}{\partial x_i}} \tag{129}$$

$$\overline{\rho}\frac{\partial k}{\partial t} + \overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \underbrace{\overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j}}_{P_k} - \underbrace{\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}}}_{\overline{\rho}\epsilon} + \underbrace{\frac{\partial \overline{u_i''t_{ij}}}{\partial x_j}}_{D} - \underbrace{\frac{\partial \overline{u_j''P'}}{\partial x_j}}_{\Pi_t} - \underbrace{\frac{1}{2}\frac{\partial \overline{\rho u_i''u_i''u_j''}}{\partial x_j}}_{T_k} - \underbrace{\overline{u_i''}\frac{\partial \overline{P}}{\partial x_i}}_{\Pi_w} + \underbrace{\overline{P'\frac{\partial u_i''}{\partial x_i}}}_{\Pi_d} \tag{130}$$

Turbulent kinetic energy per unit volume, k:

$$k = \frac{\overline{\rho u_i''u_i''}}{2\overline{\rho}} \tag{131}$$

Favre-averaged Reynolds stress tensor, $\tau_{\mathbf{ij}}$:

$$\tau_{ij} = \frac{-\overline{\rho u_i''u_j''}}{\overline{\rho}} \tag{132}$$

Turbulent kinetic energy budget terms:

- P_k: Production
- $\overline{\rho}\epsilon$: Viscous dissipation
- D: Molecular (viscous) diffusion
- Π_t: Pressure diffusion
- T_k: Turbulent transport
- Π_w: Pressure work
- Π_d: Pressure dilatation

4.6 Terms to Model

By allowing for mean density variation ($\overline{\rho}$) and density fluctuations (ρ'), which introduce density fluctuation correlations in the Favre-averaged governing equations, at a minimum four terms must be modeled in order to close the compressible-flow equations for continuity, momentum, energy, and the equation of state, namely: the Reynolds stress tensor, the turbulent heat flux, and molecular diffusion and turbulent transport of TKE. An additional four terms to model arise when considering a transport equation for the TKE itself, notable are three pressure terms and the addition of a TKE dissipation. Since TKE production and dissipation are simply transfers of energy from the mean to the turbulent fluctuation and then to internal energy, respectively, they cancel each other and are not present in the total energy equation in Eq. 92. As a reminder, the terms in the transport equations, the transport/diffusion term itself, is usually the divergence of the quantity which we model. As an example, the turbulent transport of heat is $\frac{\partial q_{Tj}}{\partial x_j}$, but we model the turbulent heat flux q_{Tj}. In this section, unless otherwise cited, the closure approximations are from the following sources:

- NASA Langley Turbulence Modeling Resource: Implementing Turbulence Models into the Compressible RANS Equations [1]
- David C. Wilcox, Turbulence Modeling for CFD 3rd Ed. [2].

The closures provided herein are the classical examples, recent developments or other "direct" or "high-order" stress closure approximations are also possible (like the full second-moment Reynolds stress models).

4.6.1 Reynolds Stress Tensor

The Reynolds stress tensor closure approximation makes use of an eddy viscosity μ_T and the Boussinesq approximation to relate the turbulent stress tensor to the mean strain rate tensor.

$$\overline{\rho}\tau_{ij} = \overline{\rho u_i'' u_j''} \approx 2\mu_T \left(\widetilde{S_{ij}} - \frac{1}{3}\frac{\partial \widetilde{u_k}}{\partial x_k}\delta_{ij} \right) - \frac{2}{3}\overline{\rho}k\delta_{ij} \tag{133}$$

$$\widetilde{S_{ij}} = \frac{1}{2}\left(\frac{\partial \widetilde{u_i}}{\partial x_j} + \frac{\partial \widetilde{u_j}}{\partial x_i} \right) \tag{134}$$

For subsonic flows, the $-\frac{2}{3}\overline{\rho}k\delta_{ij}$ is sometimes ignored. Additionally, it is required that the trace of $\tau_{ij} = -2k$; therefore, the "second eddy viscosity" must be $-\frac{2}{3}\mu_T$, similar to Stokes' hypothesis for the molecular viscosity.

4.6.2 Turbulent Heat Flux

Relating momentum and heat transfer (classical Reynolds analogy [3]), the turbulent heat flux is assumed proportional to the mean temperature gradient. The closure makes use of a turbulent Prandtl number Pr_T, which may vary spatially, but is often set to be $Pr_T \approx 0.9$ for boundary layers and $Pr_T \approx 0.5$ for free shear layers.

Turbulent transport of heat:

$$\frac{\partial q_{Tj}}{\partial x_j} \tag{135}$$

Turbulent heat flux:

$$q_{Tj} = \overline{\rho u_j'' h''} \approx -\frac{\mu_T c_p}{Pr_T}\frac{\partial \widetilde{T}}{\partial x_j} = -\frac{\mu_T}{Pr_T}\frac{\partial \widetilde{h}}{\partial x_j} \tag{136}$$

Definition of Turbulent Prandtl number without modeling, if you had data from experiment or direct numerical simulation (DNS), where direction x_1 is streamwise and x_2 is wall-normal:

$$Pr_T = \frac{\overline{\rho u_1'' u_2''}\frac{\partial \widetilde{T}}{\partial x_2}}{\overline{\rho T'' u_2''}\frac{\partial \widetilde{u_1}}{\partial x_2}} \tag{137}$$

4.6.3 Molecular diffusion and turbulent transport of turbulent kinetic energy, D and $\mathbf{T_k}$:

Molecular transport of turbulent kinetic energy:

$$D = \frac{\partial \left(\overline{t_{ij} u_i''} \right)}{\partial x_j} \tag{138}$$

Molecular diffusive flux of turbulent kinetic energy:

$$\overline{t_{ij} u_i''} \tag{139}$$

Turbulent transport of turbulent kinetic energy:

$$T_k = -\frac{1}{2} \frac{\partial \left(\overline{\rho u_j'' u_i'' u_i''} \right)}{\partial x_j} \tag{140}$$

Turbulent transport flux of turbulent kinetic energy:

$$-\frac{1}{2} \overline{\rho u_j'' u_i'' u_i''} \tag{141}$$

These two terms are often modeled together as follows:

$$\overline{t_{ij} u_i''} - \frac{1}{2} \overline{\rho u_j'' u_i'' u_i''} \approx \left(\overline{\mu} + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \tag{142}$$

where σ_k is a coefficient associated with the modeling equation for k, see [2] for the exact $k - \omega$ model coefficients. For zero-equation models and subsonic flows, these terms are often neglected.

4.6.4 Pressure diffusion, Π_t:

The pressure diffusion term acts to transport TKE like the molecular diffusion and turbulent transport terms. Therefore, pressure diffusion is often modeled/absorbed with the turbulent transport triple product.

Pressure diffusion of turbulent kinetic energy:

$$\Pi_t = -\frac{\partial \overline{u_j'' P'}}{\partial x_j} \tag{143}$$

Pressure transport flux of turbulent kinetic energy:

$$-\overline{u_j'' P'} \tag{144}$$

As mentioned previously, this term is often lumped in with the other diffusive/transport terms. By combining the turbulent transport with the pressure diffusion, the pressure diffusion is effectively ignored. Looking at just the fluxes, all three diffusive/transport terms can be modeled together as follows:

$$\overline{t_{ij} u_i''} - \frac{1}{2} \overline{\rho u_j'' u_i'' u_i''} - \overline{u_j'' P'} \approx \left(\overline{\mu} + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \tag{145}$$

4.6.5 Pressure dilatation, Π_d:

The pressure dilatation term has been shown to be small relative to the other TKE transport terms [4; 5; 2] and is often neglected for low Mach flows. However, for high-speed flow it has been shown to be important [6]. Sarkar [6] proposed the following model based on the turbulent Mach number (M_t):

$$\Pi_d = \overline{P' \frac{\partial u_i''}{\partial x_i}} \approx \alpha_2 \overline{\rho} \tau_{ij} \frac{\partial \widetilde{u}_i}{\partial x_j} M_t + \alpha_3 \overline{\rho} \epsilon M_t^2 \tag{146}$$

$$\alpha_2 = 0.15 \tag{147}$$

$$\alpha_3 = 0.2 \tag{148}$$

$$M_t = \frac{\sqrt{2k}}{\widetilde{a}} \tag{149}$$

$$\widetilde{a} = \sqrt{\gamma R_{gas} \widetilde{T}} \quad \text{(Favre-averaged speed of sound)} \tag{150}$$

where γ is the ratio of specific heats and R_{gas} is the specific gas constant. According to Wilcox [2], this model (and many other proposed models) have not received general acceptance. Pressure dilatation modeling is an active research area.

Turbulent Mach number aside:

$$M_t(x_i, t) = \frac{u_i'(x_i, t)}{\overline{a}(x_i, t)} = \frac{u_i'(x_i, t)}{\sqrt{\gamma R_{gas} \overline{T}(x_i, t)}} \quad \text{(Local, instantaneous, Reynolds form)} \tag{151}$$

$$M_t(x_i, t) = \frac{u_i''(x_i, t)}{\widetilde{a}(x_i, t)} = \frac{u_i''(x_i, t)}{\sqrt{\gamma R_{gas} \widetilde{T}(x_i, t)}} \quad \text{(Local, instantaneous, Favre form)} \tag{152}$$

$$M_t(x_i) = \frac{u'_{i,RMS}(x_i)}{\overline{a}(x_i)} = \frac{\sqrt{\overline{u_i'^2}}}{\sqrt{\gamma R_{gas} \overline{T}}} = \frac{\sqrt{\overline{u_i' u_i'}}}{\sqrt{\gamma R_{gas} \overline{T}}} \quad \text{(Time averaged, RMS, Reynolds form)} \tag{153}$$

$$M_t(x_i) = \frac{u''_{i,RMS}(x_i)}{\widetilde{a}(x_i)} = \frac{\sqrt{\widetilde{u_i''^2}}}{\sqrt{\gamma R_{gas} \widetilde{T}}} = \frac{\sqrt{\overline{\rho u_i'' u_i''}/\overline{\rho}}}{\sqrt{\gamma R_{gas} \widetilde{T}}} \quad \text{(Time averaged, RMS, Favre form)} \tag{154}$$

4.6.6 Pressure work, Π_w:

Modeling the pressure work term is primarily concerned with modeling the non-zero mean of the Favre-fluctuation for the velocity $\overline{u_i''}$.

$$\Pi_w = -\overline{u_i''} \frac{\partial \overline{P}}{\partial x_i} \tag{155}$$

The mean of the Favre-fluctuation of velocity is known as the turbulent mass flux:

$$\overline{u_i''} = \frac{-\overline{\rho' u_i'}}{\overline{\rho}} \tag{156}$$

It is possible to model the turbulent mass flux according to a gradient diffusion hypothesis (with some turbulent diffusivity D_T) [7; 8; 9]:

$$-\overline{\rho' u_i'} \approx D_T \frac{\partial \overline{\rho}}{\partial x_i} \tag{157}$$

where the turbulent diffusivity can depend on k, $\overline{\rho}\epsilon$, τ_{ij}, and M_t [2]. For perfect gas compressibility, isobaric evolution, and isothermal mixing of two non-reactive species a model for the turbulent mass flux can be obtained from the respective equations of state [9]. For the perfect gas scenario, the turbulent mass flux is linearly linked with pressure-velocity, temperature-velocity, and mass-fraction-velocity correlations via the equation of state as follows [9]:

$$\frac{\overline{\rho' u_i'}}{\overline{\rho}} \approx \frac{\overline{P' u_i'}}{\overline{P}} + \frac{\overline{T' u_i'}}{\overline{T}} \tag{158}$$

4.6.7 Viscous dissipation, $\overline{\rho}\epsilon$:

The Favre-averaged dissipation rate is as follows:

$$\overline{\rho}\epsilon = \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} = \overline{\frac{1}{2}t_{ij}\left(\frac{\partial u_i''}{\partial x_j} + \frac{\partial u_j''}{\partial x_i}\right)} = \overline{t_{ij}s_{ij}''} \tag{159}$$

where s_{ij}'' is the fluctuating strain rate tensor. When modeling, the dissipation can be split into solenoidal (ϵ_s) and dilatational (ϵ_d) components:

$$\overline{\rho}\epsilon = \overline{\rho}\epsilon_s + \overline{\rho}\epsilon_d \tag{160}$$

$$\overline{\rho}\epsilon_s = \frac{\overline{\mu}}{\overline{\rho}}\overline{\rho\omega_i''\omega_i''} \tag{161}$$

$$\overline{\rho}\epsilon_d = \frac{4}{3}\frac{\overline{\mu}}{\overline{\rho}}\overline{\rho\frac{\partial u_i''}{\partial x_i}\frac{\partial u_i''}{\partial x_i}} \tag{162}$$

where ω_i'' is the fluctuating vorticity. This decomposition relies on several assumptions as outlined by Wilcox [2] – the correlation between velocity-gradient fluctuations and kinematic viscosity fluctuations are neglected and $\frac{\partial u_i''}{\partial x_j}\frac{\partial u_j''}{\partial x_i} \approx \frac{\partial u_i''}{\partial x_i}\frac{\partial u_i''}{\partial x_i}$ is invoked to separate solenoidal and dilatational contributions. When modeling, the dissipation rate may be solved via its own transport equation, as in two-equation models such as $k-\epsilon$ or $k-\omega$, or modeled algebraically, typically as a function of k and a turbulence length/time scale. Furthermore, a distinction between a transport equation for the solenoidal or dilatational dissipation is common. ϵ_s may be treated as the primary dissipation variable and ϵ_d is modeled as a function of ϵ_s and the turbulent Mach number.

4.7 Compressible Turbulence Models

With the list of terms to model outlined and closures/model forms identified, the compressible turbulence modeling follows closely to the well-established incompressible turbulence modeling framework. Two-equation models (e.g., $k-\epsilon$ or $k-\omega$), one-equation models (e.g., Spalart-Allmaras with compressibility corrections), or algebraic zero-equation models (e.g., Prandtl's mixing-length model) can be employed. However, these models must be appropriately extended to account for compressibility effects, such as variable density, dilatational contributions to dissipation, and pressure–velocity correlations, consistent with the compressible turbulent kinetic energy budget.

The NASA Turbulence Modeling Resource [10] provides a central resource for Reynolds-averaged Navier-Stokes (RANS) turbulence model documentation.

4.8 Temperature Dependent Transport Properties

For compressible flows, not only does the density vary, but also the transport properties like molecular viscosity (μ) and molecular thermal conductivity (κ). These transport properties are considered functions of temperature. A commonly used viscosity-temperature relationship is Sutherland's law:

$$\mu(T) = \mu_{ref}\left(\frac{T}{T_{ref}}\right)^{3/2}\frac{T_{ref}+S}{T+S} \tag{163}$$

For air, $\mu_{ref} = 1.789 \times 10^{-5}$ kgm^{-1}s^{-1}, $T_{ref} = 288.2$ K, and $S = 110.4$ K [11]. Thermal conductivity is related to the molecular viscosity via the molecular (laminar) Prandtl number as in Eq. 71. Sutherland's law also assumes thermally and calorically perfect gas, i.e. chemistry cannot strictly be accommodated. A more rigorous approach to obtaining transport properties that involves solving Boltzmann equations is the Chapman-Enskog theory. Additional details may be found in Nagnibeda and Kustova [12].

5 Compressibility Transformations for the Law of the Wall

Historically, incompressible turbulent flows have been studied extensively; for an overview see the books from Pope [13] and Kajishima and Taira [14]. Extending from the body of knowledge for incompressible turbulence, computational studies concerning compressible boundary layer turbulence have predominately been concerned with assessing Morkovin's hypothesis [15], which states that differences between compressible and incompressible turbulence can be accounted for by the mean property variations, such as density and viscosity, as long as the turbulent Mach number is small and dilatation effects are negligible [16]. As a consequence, a variety of velocity and temperature scalings have been developed with the aim of collapsing the compressible statistics to their incompressible counterparts.

Note that in this section the notation has changed from the directions denoted with subscripts one through three to the classical cartesian notation of x, y, z, u, v, w to avoid the over use of subscripts and simplify the notation.

5.1 Velocity Transformations

Classically, fully turbulent, high-Reynolds number ($Re_\tau \gg 1$), no pressure gradient incompressible boundary layer flows collapse to the law of the wall when the mean velocity and wall-normal distance are normalized in inner scaling as follows:

$$y^+ = \frac{y u_\tau \overline{\rho_w}}{\overline{\mu_w}} \tag{164}$$

$$u^+ = \frac{\overline{u}}{u_\tau} \tag{165}$$

where $Re_\tau = \delta u_\tau \overline{\rho_w}/\overline{\mu_w}$ is the friction Reynolds number, δ is the boundary layer height, $u_\tau = \sqrt{\tau_w/\rho_w}$ is the friction velocity, τ_w is the wall shear stress, and subscript w denotes a 'wall' quantity. Note that in incompressible flow ρ and μ are assumed constant so $\rho_w = \rho = \overline{\rho}$ and $\mu = \mu_w = \overline{\mu}$. For compressible flow, the idea is to scale the velocity profile and wall coordinate by the mean property variation such as density and viscosity, such that you get an equivalent 'incompressible' form y_I and u_I, which then can be normalized and collapsed to the incompressible law of the wall ($y_I^+ = \frac{y_I u_\tau \overline{\rho_w}}{\overline{\mu_w}}$ and $u_I^+ = \frac{u_I}{u_\tau}$). The functional form for accounting for the mean property variation in terms of mapping functions f_I and g_I for wall distance and mean velocity, respectively, is taken from Modesti and Pirozzoli [17]:

$$y_I = \int_0^y f_I \, dy \tag{166}$$

$$u_I = \int_0^{\widetilde{u}} g_I \, d\widetilde{u} \tag{167}$$

A number of velocity transformations are available in the literature. Here six will be considered:

1. van Driest (1951) [18]

    - Compressibility effects are mainly due to mean density variation, so if a log-law structure already exists you can correct the velocity using a density weighting so that compressible data collapses onto the incompressible log-law. This is done by assuming compressible wall-bounded turbulence obeys Prandtl's incompressible mixing length assumption $\left(\frac{\partial u^+_{VD}}{\partial y^+} = (\rho^+)^{1/2}\frac{\partial \tilde{u}^+}{\partial y_+}\right)$

- Shown to work well for adiabatic walls, but breaks down when there is strong wall heating or cooling and large property variations (like viscosity or pressure gradients).

2. Zhang et al. (2012) [19]
    - Sought a velocity transformation that is Mach number invariant.
    - Replaced van Driest's mixing length assumption with the more general proposition of turbulence equilibrium, where the turbulence production and dissipation are approximately equal. This is usually only valid in the log-layer but they make the assumption for the entire inner layer.
    - The result ends up being a transformation that is viscosity weighted.
    - Like van Driest, also breaks down when there is strong wall heating or cooling (non-adiabatic).
3. Trettel and Larsson (2016) [20]
    - Derived a transformation based on a "log-law" condition to match velocity gradient in the log-layer between the raw and transformed states, and a stress balance condition to match total stress (sum of viscous plus turbulent stress equals wall shear stress) between the raw and transformed states. (i.e. log-layer scaling and near-wall momentum conservation)
    - Derived the semi-local scaling (previously presented by [21; 22]) and unified the scaling of the velocity, the Reynolds stresses, and the wall-normal coordinate.
4. Volpiani et al. (2020) [23]
    - Considered mapping functions with power-law dependence on the density and viscosity ratios: $f_I = (\rho^+)^b(\mu^+)^{-a}$ and $g_I = (\rho^+)^b(\mu^+)^{1-a}$.
    - Used DNS data to calibrate the values of the parameters a and b (goal to minimize the difference between the transformed mean velocity and incompressible reference).
    - Because this is a data-driven "fit" the model may not work for all conditions or applications.
5. Griffin et al. (2021) [24]
    - Presented a velocity transformation that is valid across the entire inner layer by using a total-stress-based balance to combine a viscous (near-wall) stress-based transformation like Trettel and Larsson [20] and a quasi-equilibrium-based transformation like Zhang et al. [19] in the log-layer. (total shear stress is the sum of the viscous and Reynolds (turbulent) shear stresses).
    - The transformation reduces to the near-wall or log-layer portions only at the locations where the assumptions underlying each transformation are valid.
6. Hasan et al. (2023) [25]
    - All the previous transformations only accounted for mean property variation.
    - This work accounts for both variable-property and intrinsic compressibility effects – extending the Trettel and Larsson [20] transformation for intrinsic compressibility effects.
    - Identified the friction Mach number $M_\tau = \frac{u_\tau}{\sqrt{\gamma R_{gas} T_w}}$ as an important transformation parameter.

The wall distance and mean velocity mappings are tabulated in Table 1 following a similar format to [17; 26]. Griffin et al. [24] and Hasan et al. [25] only provide velocity transformations – no unique wall-distance transformation like Trettel and Larsson [20]. Therefore, in Table 1, their wall distance function f_I is given as 1 (similarly presented in [26]). For plotting results, the semi-local wall-normal coordinate $y^* = \frac{y_I\sqrt{\tau_w/\rho(y)}}{\nu(y)} =$

$y^+_{TL} = \frac{y_{I,TL} u_\tau \overline{\rho_w}}{\overline{\mu_w}}$ is used, consistent with the original publications. In contrast, van Driest [18] and Zhang et al. [19] wall-normal coordinates utilize the local scaling (wall-based scaling) $y^+ = \frac{y_I u_\tau \overline{\rho_w}}{\overline{\mu_w}}$. For clarity, the full length terms shown in Table 1 are provided after the table. Effort was made to preserve the notation from the original publications where appropriate without conflict or loss of clarity.

Table 1: Velocity Transformations

Transformation	Acronym	Wall Distance f_I	Mean Velocity g_I
van Driest (1951)	VD	1	$(\rho^+)^{1/2}$
Zhang et al. (2012)	Z	1	$\frac{g_Z}{\mu^+}$
Trettel and Larsson (2016)	TL	$\frac{\partial}{\partial y}\left[\frac{y(\rho^+)^{1/2}}{\mu^+}\right]$	$\mu^+ \frac{\partial}{\partial y}\left[\frac{y(\rho^+)^{1/2}}{\mu^+}\right]$
Volpiani et al. (2020)	V	$\frac{(\rho^+)^{1/2}}{(\mu^+)^{3/2}}$	$\frac{(\rho^+)^{1/2}}{(\mu^+)^{1/2}}$
Griffin et al. (2021)	G	1	$S_t^+ \frac{dy^*}{d\widetilde{u}^+}$
Hasan et al. (2023)	H	1	$\left(\frac{1+\kappa y^* D^c}{1+\kappa y^* D^i}\right)\left(1 - \frac{y}{\delta_v^*}\frac{d\delta_v^*}{dy}\right)(\rho^+)^{1/2}$

$$\rho^+ = \frac{\overline{\rho}}{\overline{\rho_w}} \tag{168}$$

$$\mu^+ = \frac{\overline{\mu}}{\overline{\mu_w}} \tag{169}$$

$$S_Z = \frac{1}{\mu^+}\frac{\partial \widetilde{u}^+}{\partial y^+} \tag{170}$$

$$g_Z = \frac{-\frac{S_z}{2} + \left(\left(\frac{S_Z}{2}\right)^2 + 1 - (\mu^+)^2 S_Z\right)^{1/2}}{1 - (\mu^+)^2 S_Z} \tag{171}$$

$$S^+_{eq} = \frac{1}{\mu^+}\frac{\partial \widetilde{u}^+}{\partial y^*} \tag{172}$$

$$S^+_{TL} = \mu^+ \frac{\partial \widetilde{u}^+}{\partial y^*} \tag{173}$$

$$\tau^+ = \tau^+_{visc} + \tau^+_{turb} \tag{174}$$

$$S^+_t = \frac{\tau^+ S^+_{eq}}{\tau^+ + S^+_{eq} - S^+_{TL}} \tag{175}$$

$$\kappa = 0.41 \quad \text{(von Kármán constant)} \tag{176}$$

$$A^+ = 17 \tag{177}$$

$$f(M_\tau) = 19.3 M_\tau \tag{178}$$

$$M_\tau = \frac{u_\tau}{\sqrt{\gamma R_{gas} T_w}} \tag{179}$$

$$u^*_\tau = \sqrt{\frac{\tau_w}{\overline{\rho}(y)}} \tag{180}$$

$$\delta^*_v = \frac{\overline{\mu}(y)}{\overline{\rho}(y) u^*_\tau} \tag{181}$$

$$D^i = \left[1 - \exp\left(-\frac{y^*}{A^+}\right)\right]^2 \tag{182}$$

$$D^c = \left[1 - \exp\left(-\frac{y^*}{A^+ + f(M_\tau)}\right)\right]^2 \tag{183}$$

γ is the ratio of specific heats and R_{gas} is the specific gas constant.

5.2 Temperature Transformations

Similarly to the momentum (velocity) boundary layer, the thermal (temperature) boundary layer can also be scaled in inner units. Early work addressing temperature scalings are Kader [27] and Bradshaw and Huang [28]. For low-speed flows, conventional mean scalar scaling often assumes the scalar to be a passive admixture that does not influence the flow dynamics [27]. Relevant scalars are temperature or concentration. For the temperature analysis a mean temperature difference is defined as follows:

$$\overline{\theta} = \overline{T_w} - \overline{T} \tag{184}$$

$$\widetilde{\theta} = \widetilde{T_w} - \widetilde{T} \tag{185}$$

For very cold walls you can get a negative temperature difference and a negative wall heat flux $q_w = -\kappa \frac{\partial T}{\partial y}$. Ensuring consistent sign convention is necessary to for the negatives to cancel in the normalization. A friction temperature analogous to the friction velocity is based on the wall heat flux q_w:

$$\theta_\tau = \frac{\overline{q_w}}{\overline{\rho_w} c_p u_\tau} \tag{186}$$

where c_p is the specific heat at constant pressure. Normalizing the mean temperature difference by the friction temperature results in the conventional scaling, where the normalized wall-normal distance is the same as Eq. 164:

$$\theta^+ = \frac{\overline{\theta}}{\theta_\tau} \tag{187}$$

For incompressible flow, the density is constant and the wall subscript, w, is redundant. Eq. 187 presents a couple of challenges for compressible flows, namely:

1. This scaling does not account for mean property variations.
2. The friction temperature normalization is undefined for adiabatic flow that has zero wall heat flux.

The first issue has been addressed by mimicking the idea of an equivalent incompressible mapping, as seen in Eq. 167 ($\theta^+ = \theta_I/\theta_\tau$). van Driest type or semi-local type transformed temperatures are common place, and a recent number of temperature transformations have been proposed and assessed [29; 30; 31; 32; 33; 34].

$$\theta_I = \int_0^{\widetilde{\theta}} h_I \, d\widetilde{\theta} \quad \text{or} \quad \theta^+_{trans} = \int_0^{\theta^+} h_I \, d\theta^+ \tag{188}$$

The second issue has only recently received attention. Recent efforts by Chen et al. [32] and Zhang et al. [34] have been to define a friction temperature that accounts for the diffusive flux from the Favre-averaged energy equation in addition to the wall heat flux, such that the temperature transformation applies for both isothermal and adiabatic walls. A list of the temperature transformations presently considered is provided in Table 2.

Table 2: Temperature Transformations

Transformation	Acronym	Wall Distance f_I	Mean Temperature h_I
van Driest - type Patel et al. (2017)	VD	1	$\left(\rho^+\right)^{1/2}$
semi-local - type Patel et al. (2017)	SL	1	$\left(\rho^+\right)^{1/2}\left[1+\frac{y}{Re_\tau^*}\frac{dRe_\tau^*}{dy}\right]$
van Driest - type Chen et al. (2022)	VDc	1	$\left(\theta^*_{\tau,c}\right)^{-1}$
semi-local - type Chen et al. (2022)	SLc	1	$\left(\theta^*_{\tau,c}\right)^{-1}\left[1+\frac{y}{Re_\tau^*}\frac{dRe_\tau^*}{dy}\right]$

In Table 2 no unique wall-distance transformation is provided, therefore the wall distance mapping function is set at one. Nevertheless, when plotting, the local (y^+) or semi-local (y^*) wall-normal coordinate should be used appropriately. The last two rows in Table 2 with acronyms 'c' are 'corrected' terms to allow for the adiabatic condition. They follow a slightly different mapping notation than Eq. 188 because they remove any explicit dependence on the wall heat flux only friction temperature:

$$\theta^+_{trans,c} = \int_0^{\widetilde{\theta}} h_I \, d\widetilde{\theta} \tag{189}$$

as opposed to the isothermal only mappings (first two rows):

$$\theta_I = \int_0^{\widetilde{\theta}} h_I \, d\widetilde{\theta} \tag{190}$$

$$\theta^+_{trans} = \frac{\theta_I}{\theta_\tau} \tag{191}$$

The full definition of the various components in Table 2 are as follows:

$$\rho^+ = \frac{\overline{\rho}}{\overline{\rho_w}} \tag{192}$$

$$Re^*_\tau = Re_\tau \sqrt{\frac{\overline{\rho}}{\overline{\rho_w}}}\frac{\overline{\mu_w}}{\overline{\mu}} \tag{193}$$

$$u^*_\tau = \sqrt{\frac{\overline{\tau_w}}{\overline{\rho}}} \tag{194}$$

$$\theta^*_\tau = \frac{\overline{q_w}}{\overline{\rho} c_p u^*_\tau} \tag{195}$$

$$\theta^*_{\tau,c} = \frac{\overline{q_w} + \overline{q}}{\overline{\rho} c_p u^*_\tau} \tag{196}$$

$$\overline{q} = \overline{t_{i2} u_i} - \overline{\rho v'' u_i''}\widetilde{u}_i - \frac{\overline{\rho v'' u_i'' u_i''}}{2} \tag{197}$$

$\overline{q}$ collects all the diffusion (transport) terms of the kinetic energy, both mean and turbulent, from the Favre-averaged energy equation, Eq. 92. Moreover, it makes use of the assumptions that the flow is steady and the boundary layer is thin, where the boundary layer thickness $\delta(x) \ll x$ such that $\overline{u_2} \ll \overline{u_1}$ and $\frac{\partial}{\partial x_1} \ll \frac{\partial}{\partial x_2}$.

Unlike velocity profiles that generally follow a similar wake/defect form in the outer layer, the temperature difference θ in the outer layer may have large variation depending on the wall temperature relative to the freestream temperature ($T_w - T_\infty$), causing the θ^+ profiles to diverge in the outer layer. Additionally, $q_w + \overline{q}$ may be zero (or sign change) near the outer layer leading to a 'divide-by-zero' situation, potentially leading to an unbounded value of the normalized θ^+. Consequently, θ^+ profiles are occasionally truncated shortly after reaching log-layer.

5.2.1 Thermal Sublayer

In the near-wall region, analogous to the viscous sublayer, a thermal sublayer can be defined. It is also known as the conductive or molecular transport sublayer because near the wall the molecular conduction dominates the heat transfer. It depends on the Prandtl number at the wall as follows:

$$\theta^+ = Pr_w y^+ \tag{198}$$

$$\text{or} \tag{199}$$

$$\theta^+ = Pr_w y^* \tag{200}$$

Using the correct local or semi-local wall-normal coordinate depending on the mean temperature mapping. The molecular Prandtl number at the wall is $Pr_w = \frac{c_p \overline{\mu_w}}{\overline{\kappa_w}}$ (caution with the repeated use of notation for κ being both the thermal conductivity and von Kármán constant).

5.2.2 Thermal Log-Layer

In the vicinity of the velocity boundary layer's logarithmic layer, the temperature distribution is expected to also follow a logarithmic profile. The logarithmic scaling of the mean velocity and the Reynolds analogy suggest a logarithmic scaling of the temperature of the form [27]:

$$\theta^+ = \frac{1}{\kappa_T} \ln y^+ + B(Pr) \tag{201}$$

$$\text{or} \tag{202}$$

$$\theta^+ = \frac{1}{\kappa_T} \ln y^* + B(Pr) \tag{203}$$

Using the correct local or semi-local wall-normal coordinate depending on the mean temperature mapping. $\kappa_T \approx 0.47$ is a thermal counterpart to the von Kármán constant $\kappa \approx 0.41$. If a turbulent Prandtl number of $Pr_t \approx 0.87$ is used, $1/\kappa_T \approx Pr_t/\kappa$. For compressible flow, the offset B is a function of the Prandtl number. In contemporary studies, the semi-local-scaled molecular Prandtl number $Pr^* = \frac{c_p \overline{\mu}}{\overline{\kappa}}$ is used for the function B, as opposed to the molecular Prandtl number at the wall (in the definition of Pr, κ is the thermal conductivity, not to be confused with the von Kármán constant κ in the log-law).

5.2.3 Kader's Relation

Combining the various regions of the boundary layer for a scalar like temperature, Kader [27] suggested the following equation:

$$\theta^+ = Pr y^+ \exp(-\Gamma) + \left\{ 2.12 \ln \left[(1+y^+) \frac{1.5(2-y/R)}{1+2(1-y/R)^2} \right] + B(Pr) \right\} \exp(-1/\Gamma) \tag{204}$$

for a tube or a channel and

$$\theta^+ = Pr y^+ \exp(-\Gamma) + \left\{ 2.12 \ln \left[(1+y^+) \frac{2.5(2-y/\delta)}{1+4(1-y/\delta)^2} \right] + B(Pr) \right\} \exp(-1/\Gamma) \tag{205}$$

for a boundary layer.

The function $B(Pr)$ that determines the temperature difference between the wall and the lower edge of the logarithmic layer was based on experimental data at the time.

$$B(Pr) = (3.85 Pr^{1/3} - 1.3)^2 + 2.12 \ln Pr \tag{206}$$

Lastly Γ is as follows:

$$\Gamma = \frac{10^{-2}(Pr y^+)^4}{1 + 5 Pr^3 y^+} \tag{207}$$

Lee et al. [35] adapt Kader's formulation to include local variations, making use of semi-local values. Showing the boundary layer form:

$$\theta^+ = Pr^* y^* \exp(-\Gamma) + \left\{ 2.12 \ln \left[(1+y^*) \frac{2.5(2-y/\delta)}{1+4(1-y/\delta)^2} \right] + B(Pr_v) \right\} \exp(-1/\Gamma) \tag{208}$$

$$B(Pr_v) = (3.85 Pr_v^{1/3} - 1.3)^2 + 2.12 \ln Pr_v \tag{209}$$

$$Pr_v = Pr^*(y^* \approx 30) \tag{210}$$

$$\Gamma = \frac{10^{-2}(Pr^* y^*)^4}{1 + 5 {Pr^*}^3 y^*} \tag{211}$$

6 Velocity-Temperature Relationships

Contrary to the previous section regarding the velocity and temperature transformations, in this section we return to the subscript index notation for the streamwise, wall-normal, and spanwise coordinates (x_1, x_2, x_3) and (u_1, u_2, u_3).

6.1 Mean Flow

It has been shown that the mean temperature is proportional to the square of the mean streamwise velocity. In the context of laminar boundary layers Busemann [36] and Crocco [37] independently derived:

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \frac{T_{c,\infty} - T_w}{T_\infty} \frac{u_1}{U_\infty} + \frac{T_\infty - T_{c,\infty}}{T_\infty} \left(\frac{u_1}{U_\infty} \right)^2 \tag{212}$$

$$T_{c,\infty} = T_\infty + c \frac{U_\infty^2}{2 c_p} \tag{213}$$

where $c = 1$ in the original Crocco-Busemann relation. Later that factor was modified to the recovery factor r by Walz [38]. Resulting in Eq. 212 being equivalent to Eq. 214 if c is set to the recovery factor r. U_∞, T_∞, M_∞ are the freestream velocity, temperature, and Mach number respectively, and γ is the ratio of specific heats.

$$\frac{T}{T_\infty} = 1 + \frac{T_r - T_w}{T_\infty} \left(\frac{u_1}{U_\infty} - 1 \right) + r \frac{\gamma - 1}{2} M_\infty^2 \left(1 - \left(\frac{u_1}{U_\infty} \right)^2 \right) \tag{214}$$

The recovery temperature is defined as follows, where $r \approx 0.9$ for a boundary layer:

$$T_r = T_\infty \left(1 + r \frac{\gamma - 1}{2} M_\infty^2 \right) \tag{215}$$

$$T_r = T_\infty + r \frac{U_\infty^2}{2 c_p} \tag{216}$$

Presently, these equations are written in terms of instantaneous values; however, they are applicable for mean temperatures and velocities. Although the Walz relation improves upon the Crocco-Busemann relation for non-adiabatic flows, recent work has focused on further modifying these "quadratic" velocity-temperature relationships. To account for the effects where $Pr \neq 1$ and diabatic walls, Zhang et al. [39] developed a generalized Reynolds analogy (GRA) with the same quadratic form as the Crocco–Busemann relation and Walz's equation, but adopting the general recovery factor r_g. They introduce a generalized analogy between the total enthalpy and streamwise velocity $\overline{H_g} - \overline{H_w} = U_w \overline{u_1}$, where $H_g = c_p T + r_g u_1^2/2$ and $U_w = -Pr \overline{q_w} / \overline{\tau_w}$.

A key assumption in their work is that the effective turbulent Prandtl number is constant and equal one, $\overline{Pr_e} \approx 1$.

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \frac{T_r - T_w}{T_\infty} f\left(\frac{u_1}{U_\infty}\right) + \frac{T_\infty - T_r}{T_\infty}\left(\frac{u_1}{U_\infty}\right)^2 \tag{217}$$

$$f\left(\frac{u_1}{U_\infty}\right) = (1 - sPr)\left(\frac{u_1}{U_\infty}\right)^2 + sPr\left(\frac{u_1}{U_\infty}\right) \tag{218}$$

$$r_g = \frac{T_w - T_\infty}{U_\infty^2/(2c_p)} - \frac{2Pr}{U_\infty}\frac{q_w}{\tau_w} = r\left[sPr + (1 - sPr)\Theta\right] \quad (\text{if } \overline{Pr_e} = 1) \tag{219}$$

$$\Theta = \frac{T_w - T_\infty}{T_r - T_\infty} \tag{220}$$

$$s \equiv \frac{2C_h}{C_f} = \frac{q_w U_\infty}{\tau_w c_p (T_w - T_r)} \quad (\text{Reynolds analogy factor}) \tag{221}$$

where $C_h = \frac{q_w}{\rho_\infty U_\infty c_p (T_w - T_r)}$ is the Stanton number and $C_f = \frac{2\tau_w}{\rho_\infty U_\infty^2}$ is the skin friction coefficient.

6.2 Fluctuations

Based on the similarity between the momentum and energy equation (when written in total enthalpy form), relationships also exist between the correlations of streamwise velocity fluctuations with temperature fluctuations. First identified by Morkovin [15], a set of velocity-temperature fluctuation correlations referred to as the strong Reynolds analogy (SRA) arises from a 'strong' analogy between total enthalpy and velocity: $H' = U_w u_1'$, where U_w is a proportionality constant with dimension of velocity as shown in the previous section. One of the most common forms (from the collective set of equations identified by Morkovin [15]) is a root-mean-square form as follows:

$$\frac{\frac{\sqrt{\overline{T'^2}}}{\overline{T}}}{(\gamma - 1)M^2 \frac{\sqrt{\overline{u_1'^2}}}{\overline{u_1}}} = 1 \tag{222}$$

Numerous modified forms of the SRA have been proposed over the years, most taking the general form:

$$\frac{\frac{\sqrt{\overline{T'^2}}}{\overline{T}}}{(\gamma - 1)M^2 \frac{\sqrt{\overline{u_1'^2}}}{\overline{u_1}}} = \frac{1}{a\left|1 - \frac{\partial \overline{T_t}}{\partial \overline{T}}\right|} \tag{223}$$

where $T_t = T + \frac{u_1^2}{2c_p}$ is the total temperature and a is a ratio between velocity mixing length $\sqrt{\overline{u_1'^2}}\frac{\partial \overline{u_1}}{\partial y}$ and the temperature mixing length $\sqrt{\overline{T'^2}}\frac{\partial \overline{T}}{\partial y}$. An absolute value is used in the denominator to ensure the root-mean-square form takes a positive value regardless of the sign of total temperature gradient term. Table 3 lists different choices for a from Gaviglio [40] (GSRA), Rubesin [41] (RSRA), Huang et al. [21] (HSRA), and Zhang et al. [39] who suggest a modified turbulent Prandtl number for the HSRA (MHRSA). The SRA row is N/A because it does not take the modified form with the total temperature denominator excluded based on the assumptions that $Pr = 1$ and $T_t' = 0$.

Table 3: Modified Strong Reynolds Analogy Parameters

Reference	Acronym	a
Morkovin (1962)	SRA	N/A
Gaviglio (1987)	GSRA	1
Rubesin (1990)	RSRA	1.34
Huang et al. (1995)	HSRA	Pr_t
Zhang et al. (2014)	MHRSA	$\overline{Pr_t}$

The GRA from Zhang et al. [39] was also shown for the fluctuation correlations. In fact the GRA gets is name by 'generalizing' the SRA to have a generalized recovery enthalpy that depends on a generalized recovery factor: $H_g' + c_p\phi' = U_w u_1'$, where $H_g = c_p T + r_g \frac{u_1^2}{2}$. Note that the SRA and GRA total enthalpy and streamwise velocity relationships hold for the mean flow, even though they are shown presently for the fluctuations. In root-mean-square form the GRA is as follows:

$$\sqrt{\overline{(T'+\phi')^2}} = \frac{1}{\overline{Pr_e}} \left| \frac{\partial \overline{T}}{\partial \overline{u_1}} \right| \sqrt{\overline{u_1'^2}} \tag{224}$$

$$\sqrt{\overline{(T'+\phi')^2}} = \left| \frac{\partial \overline{T}}{\partial \overline{u_1}} \right| \sqrt{\overline{u_1'^2}} \quad (\text{if } \overline{Pr_e} = 1) \tag{225}$$

The instantaneous definition gives insight into how the temperature fluctuations are not directly tied to the velocity fluctuations. The first term on the right-hand-side of Eq. 226 suggests temperature fluctuations are in part due to the velocity fluctuations; however, they define an additional residual temperature fluctuation ϕ' that encompass the features that break the SRA etc.

$$T' = \frac{1}{\overline{Pr_e}} \frac{\partial \overline{T}}{\partial \overline{u_1}} u_1' - \phi' \tag{226}$$

An aside on the effective Prandtl number is included below, due to its importance in setting up the GRA:

$$\overline{Pr_e} = \frac{\overline{Pr_t}}{1+\epsilon} \approx 1 \tag{227}$$

$$\overline{Pr_t} = \frac{\overline{(\rho u_2)' u_1'}}{\overline{(\rho u_2)' T'}} \frac{\frac{\partial \overline{T}}{\partial y}}{\frac{\partial \overline{u_1}}{\partial y}} = Pr_T \frac{1 + \overline{u_2}\,\overline{\rho' u_1'} / \overline{\rho u_2' u_1'}}{1 + \overline{u_2}\,\overline{\rho' T'} / \overline{\rho u_2' T'}} \tag{228}$$

$$Pr_T = \frac{\overline{\rho u_2' u_1'}\, \frac{\partial \overline{T}}{\partial x_2}}{\overline{\rho u_2' T'}\, \frac{\partial \overline{u_1}}{\partial x_2}} \tag{229}$$

$$\epsilon = \frac{\overline{(\rho u_2)' \phi'}}{\overline{(\rho u_2)' T'}} \tag{230}$$

$$\phi' = -\left(\frac{Pr \overline{q_w}}{c_p \overline{\tau_w}} + \frac{r_g \overline{u_1}}{c_p} \right) u_1' - T' \tag{231}$$

$$r_g = \frac{2c_p}{\overline{u_1}^2} \left(\overline{T_w} - \overline{T} - \frac{Pr \overline{q_w}}{c_p \overline{\tau_w}} \overline{u_1} \right) \tag{232}$$

Substituting the above equations, Eq. 227 to 232, into Eq. 225 results in a simplified equation for the GRA that depends purely on mean quantities:

$$\frac{|A|}{\left| \frac{\partial \overline{T}}{\partial \overline{u_1}} \right|} = 1 \tag{233}$$

$$|A| = \left| \frac{2}{c_p} \left(\overline{T_w} - \overline{T} \right) - \frac{Pr \overline{q_w}}{c_p \overline{\tau_w}} \right| = \left| \frac{2\overline{\theta}}{c_p} - \frac{Pr \overline{q_w}}{c_p \overline{\tau_w}} \right| \tag{234}$$

It is worth noting that the sign of the relationship (correlation) between the velocity and temperature fluctuations may be positive or negative. Ejection and sweep events advect fluid across the mean temperature field, and their net contribution depends on the relative thermal state of the near-wall and outer flow (e.g., wall heating or cooling).

In general, all means and fluctuations listed in this section should make use of the Favre decomposition even though they are not listed with a double quote $(\cdot)''$ or tilde $\widetilde{(\cdot)}$ explicitly in the literature and present formulations.

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