1 Introduction

The aim of this document is to decompose the turbulent kinetic energy (TKE) budget and heat transfer terms so that they are expressed in simplified forms for statistics collection. The use case is for the output of a computational fluid dynamics (CFD) code, specifically scale resolving simulations like direct numerical simulations (DNS). The goal of each simplification is to express the result in terms of simple Reynolds averages of instantaneous values or products of instantaneous values. Moreover, averaging of first order spatial derivatives is assumed to be available from the CFD code (i.e. $\overline{\frac{\partial u_i}{\partial x_j}}$ is available), for the primitives ρ, u_1, u_2, u_3, and T only. Preexisting knowledge of the TKE budget derivation and turbulence in compressible flows is expected and will not be discussed in detail presently. Standard nomenclature is used throughout this document. Unless otherwise noted, index (summation) notation is utilized where x_1, x_2, and x_3 correspond the streamwise, wall-normal, and spanwise directions, receptively. Likewise u_1, u_2, and u_3 correspond the streamwise, wall-normal, and spanwise velocities. In terms of thermodynamic variables, P is static pressure, ρ is static density, T is static temperature, and μ is dynamic viscoisty.

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2 TKE Transport Equation

The transport equation for turbulent kinetic energy is as follows:

$$\overline{\rho}\frac{\partial k}{\partial t} + \overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} - \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} + \frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} - \frac{\partial \overline{u_j'' P'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j} - \overline{u_i''}\frac{\partial \overline{P}}{\partial x_i} + \overline{P'\frac{\partial u_i''}{\partial x_i}} \tag{1}$$

Assuming steady flow and moving the advection term to the right-hand-side, the budget with zero sum is the following:

$$0 = \underbrace{-\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j}}_{A} + \underbrace{\overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j}}_{P_k} - \underbrace{\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}}}_{\overline{\rho}\epsilon} + \underbrace{\frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j}}_{D} - \underbrace{\frac{\partial \overline{u_j'' P'}}{\partial x_j}}_{\Pi_t} - \underbrace{\frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j}}_{T_k} - \underbrace{\overline{u_i''}\frac{\partial \overline{P}}{\partial x_i}}_{\Pi_w} + \underbrace{\overline{P'\frac{\partial u_i''}{\partial x_i}}}_{\Pi_d} \tag{2}$$

Turbulent kinetic energy per unit volume, k:

$$k = \frac{\overline{\rho u_i'' u_i''}}{2\overline{\rho}} \tag{3}$$

Favre-averaged Reynolds stress tensor, τ_{ij}:

$$\tau_{ij} = \frac{-\overline{\rho u_i'' u_j''}}{\overline{\rho}} \tag{4}$$

Viscous stress tensor, t_{ij}:

$$t_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{5}$$

$$t_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij} \quad \text{(Stokes's hypothesis)} \tag{6}$$

where λ is the second coefficient of viscosity and δ_{ij} is the Kronecker delta.

Turbulent kinetic energy budget terms:

- A: Advection
- P_k: Production
- $\overline{\rho}\epsilon$: Viscous dissipation
- D: Molecular (viscous) diffusion
- Π_t: Pressure diffusion
- T_k: Turbulent transport
- Π_w: Pressure work
- Π_d: Pressure dilatation

3 Useful Simplifications

Gradient of the product of two Favre-fluctuating quantities, $\frac{\partial\overline{\rho\phi''\psi''}}{\partial x_k}$:

The gradient of the product of Favre-fluctuating velocities arises frequently, for clarity the simplification of the gradient of the product of two Favre-fluctuating quantities is shown here with arbitrary parameters ϕ and ψ:

$$\frac{\partial\overline{\rho\phi''\psi''}}{\partial x_k} = \overline{\frac{\partial}{\partial x_k}\left[\rho\phi''\psi''\right]} \tag{7}$$

To find $\frac{\partial\overline{\rho\phi''\psi''}}{\partial x_k}$, let's first look at $\frac{\partial\overline{\rho\phi\psi}}{\partial x_k}$.

$$\frac{\partial\overline{\rho\phi\psi}}{\partial x_k} = \overline{\frac{\partial}{\partial x_k}\left[\rho\phi\psi\right]} \tag{8}$$

$$\frac{\partial\overline{\rho\phi\psi}}{\partial x_k} = \overline{\frac{\partial}{\partial x_k}\left[\rho(\widetilde{\phi}+\phi'')(\widetilde{\psi}+\psi'')\right]} \tag{9}$$

$$\frac{\partial\overline{\rho\phi\psi}}{\partial x_k} = \overline{\frac{\partial}{\partial x_k}\left[\rho\widetilde{\phi}\widetilde{\psi}+\rho\widetilde{\phi}\psi''+\rho\phi''\widetilde{\psi}+\rho\phi''\psi''\right]} \tag{10}$$

$$\frac{\partial\overline{\rho\phi\psi}}{\partial x_k} = \frac{\partial}{\partial x_k}\left[\overline{\rho\widetilde{\phi}\widetilde{\psi}}+\overline{\rho\widetilde{\phi}\psi''}+\overline{\rho\phi''\widetilde{\psi}}+\overline{\rho\phi''\psi''}\right] \tag{11}$$

$$\frac{\partial\overline{\rho\phi\psi}}{\partial x_k} = \frac{\partial}{\partial x_k}\left[\overline{\rho}\widetilde{\phi}\widetilde{\psi}+\cancelto{0}{\overline{\rho\psi''}\widetilde{\phi}}+\cancelto{0}{\overline{\rho\phi''}\widetilde{\psi}}+\overline{\rho\phi''\psi''}\right] \tag{12}$$

$$\frac{\partial\overline{\rho\phi\psi}}{\partial x_k} = \frac{\partial}{\partial x_k}\left[\overline{\rho}\widetilde{\phi}\widetilde{\psi}\right]+\frac{\partial}{\partial x_k}\left[\overline{\rho\phi''\psi''}\right] \tag{13}$$

$$\frac{\partial\overline{\rho\phi\psi}}{\partial x_k} = \frac{\partial\overline{\rho}}{\partial x_k}\widetilde{\phi}\widetilde{\psi}+\frac{\partial\widetilde{\phi}}{\partial x_k}\overline{\rho}\widetilde{\psi}+\frac{\partial\widetilde{\psi}}{\partial x_k}\overline{\rho}\widetilde{\phi}+\frac{\partial\overline{\rho\phi''\psi''}}{\partial x_k} \tag{14}$$

$$\frac{\partial\overline{\rho\phi\psi}}{\partial x_k} = \overline{\frac{\partial\rho}{\partial x_k}}\widetilde{\phi}\widetilde{\psi}+\frac{\partial\widetilde{\phi}}{\partial x_k}\overline{\rho}\widetilde{\psi}+\frac{\partial\widetilde{\psi}}{\partial x_k}\overline{\rho}\widetilde{\phi}+\frac{\partial\overline{\rho\phi''\psi''}}{\partial x_k} \tag{15}$$

Rearranging for the product of the fluctuations:

$$\frac{\partial \overline{\rho\phi''\psi''}}{\partial x_k} = \frac{\partial \overline{\rho\phi\psi}}{\partial x_k} - \overline{\frac{\partial \rho}{\partial x_k}}\widetilde{\phi}\widetilde{\psi} - \frac{\partial \widetilde{\phi}}{\partial x_k}\overline{\rho}\widetilde{\psi} - \frac{\partial \widetilde{\psi}}{\partial x_k}\overline{\rho}\widetilde{\phi} \tag{16}$$

$$\frac{\partial \overline{\rho\phi''\psi''}}{\partial x_k} = \overline{\frac{\partial \rho\phi\psi}{\partial x_k}} - \overline{\frac{\partial \rho}{\partial x_k}}\widetilde{\phi}\widetilde{\psi} - \frac{\partial \widetilde{\phi}}{\partial x_k}\overline{\rho}\widetilde{\psi} - \frac{\partial \widetilde{\psi}}{\partial x_k}\overline{\rho}\widetilde{\phi} \tag{17}$$

$$\tag{18}$$

Expanding $\overline{\frac{\partial \rho\phi\psi}{\partial x_k}}$:

$$\frac{\partial \overline{\rho\phi''\psi''}}{\partial x_k} = \overline{\frac{\partial \rho}{\partial x_k}\phi\psi} + \overline{\frac{\partial \phi}{\partial x_k}\rho\psi} + \overline{\frac{\partial \psi}{\partial x_k}\rho\phi} - \overline{\frac{\partial \rho}{\partial x_k}}\widetilde{\phi}\widetilde{\psi} - \frac{\partial \widetilde{\phi}}{\partial x_k}\overline{\rho}\widetilde{\psi} - \frac{\partial \widetilde{\psi}}{\partial x_k}\overline{\rho}\widetilde{\phi} \tag{19}$$

Gradient of a Favre mean quantity, $\frac{\partial \widetilde{\phi}}{\partial \mathbf{x_k}}$:

$$\frac{\partial \widetilde{\phi}}{\partial x_k} = \frac{\partial \left[\overline{\rho\phi}/\overline{\rho}\right]}{\partial x_k} \tag{20}$$

$$\frac{\partial \widetilde{\phi}}{\partial x_k} = \frac{1}{\overline{\rho}^2}\left[\frac{\partial \overline{\rho\phi}}{\partial x_k}\overline{\rho} - \frac{\partial \overline{\rho}}{\partial x_k}\overline{\rho\phi}\right] \tag{21}$$

$$\frac{\partial \widetilde{\phi}}{\partial x_k} = \frac{1}{\overline{\rho}^2}\left[\overline{\frac{\partial \rho\phi}{\partial x_k}}\overline{\rho} - \overline{\frac{\partial \rho}{\partial x_k}}\overline{\rho\phi}\right] \tag{22}$$

$$\frac{\partial \widetilde{\phi}}{\partial x_k} = \frac{1}{\overline{\rho}^2}\left[\overline{\left(\frac{\partial \rho}{\partial x_k}\phi + \frac{\partial \phi}{\partial x_k}\rho\right)}\overline{\rho} - \overline{\frac{\partial \rho}{\partial x_k}}\overline{\rho\phi}\right] \tag{23}$$

$$\frac{\partial \widetilde{\phi}}{\partial x_k} = \frac{1}{\overline{\rho}^2}\left[\left(\overline{\frac{\partial \rho}{\partial x_k}\phi} + \overline{\frac{\partial \phi}{\partial x_k}\rho}\right)\overline{\rho} - \overline{\frac{\partial \rho}{\partial x_k}}\overline{\rho\phi}\right] \tag{24}$$

$$\frac{\partial \widetilde{\phi}}{\partial x_k} = \frac{1}{\overline{\rho}}\left(\overline{\frac{\partial \rho}{\partial x_k}\phi} + \overline{\frac{\partial \phi}{\partial x_k}\rho}\right) - \frac{1}{\overline{\rho}^2}\overline{\frac{\partial \rho}{\partial x_k}}\overline{\rho\phi} \tag{25}$$

Product of two Favre-fluctuating quantities, $\overline{\rho\phi''\psi''}$:
To find $\overline{\rho\phi''\psi''}$, let's first look at $\overline{\rho\phi\psi}$.

$$\overline{\rho\phi\psi} = \overline{\rho(\widetilde{\phi} + \phi'')(\widetilde{\psi} + \psi'')} \tag{26}$$

$$\overline{\rho\phi\psi} = \overline{\rho\widetilde{\phi}\widetilde{\psi} + \rho\widetilde{\phi}\psi'' + \rho\phi''\widetilde{\psi} + \rho\phi''\psi''} \tag{27}$$

$$\overline{\rho\phi\psi} = \overline{\rho\widetilde{\phi}\widetilde{\psi} + \rho\widetilde{\phi}\psi'' + \rho\phi''\widetilde{\psi} + \rho\phi''\psi''} \tag{28}$$

$$\overline{\rho\phi\psi} = \overline{\rho\widetilde{\phi}\widetilde{\psi}} + \cancelto{0}{\overline{\rho\psi''}}\widetilde{\phi} + \cancelto{0}{\overline{\rho\phi''}}\widetilde{\psi} + \overline{\rho\phi''\psi''} \tag{29}$$

$$\overline{\rho\phi\psi} = \overline{\rho}\widetilde{\phi}\widetilde{\psi} + \overline{\rho\phi''\psi''} \tag{30}$$

Rearranging for the product of the fluctuations:

$$\overline{\rho\phi''\psi''} = \overline{\rho\phi\psi} - \overline{\rho}\widetilde{\phi}\widetilde{\psi} \tag{31}$$

Product of three Favre-fluctuating quantities, $\overline{\rho\phi''\psi''\chi''}$:
In addition to ϕ and ψ, an additional arbitrary parameter χ is introduced.

$$\overline{\rho\phi''\psi''\chi''} = \overline{\rho(\phi-\widetilde{\phi})(\psi-\widetilde{\psi})(\chi-\widetilde{\chi})} \tag{32}$$

$$\overline{\rho\phi''\psi''\chi''} = \overline{\rho(\phi\psi-\phi\widetilde{\psi}-\widetilde{\phi}\psi+\widetilde{\phi}\widetilde{\psi})(\chi-\widetilde{\chi})} \tag{33}$$

$$\overline{\rho\phi''\psi''\chi''} = \overline{\rho(\phi\psi\chi-\phi\widetilde{\psi}\chi-\widetilde{\phi}\psi\chi+\widetilde{\phi}\widetilde{\psi}\chi-\phi\psi\widetilde{\chi}+\phi\widetilde{\psi}\widetilde{\chi}+\widetilde{\phi}\psi\widetilde{\chi}-\widetilde{\phi}\widetilde{\psi}\widetilde{\chi})} \tag{34}$$

$$\overline{\rho\phi''\psi''\chi''} = \overline{\rho\phi\psi\chi}-\overline{\rho\phi\widetilde{\psi}\chi}-\overline{\rho\widetilde{\phi}\psi\chi}+\overline{\rho\widetilde{\phi}\widetilde{\psi}\chi}-\overline{\rho\phi\psi\widetilde{\chi}}+\overline{\rho\phi\widetilde{\psi}\widetilde{\chi}}+\overline{\rho\widetilde{\phi}\psi\widetilde{\chi}}-\overline{\rho\widetilde{\phi}\widetilde{\psi}\widetilde{\chi}} \tag{35}$$

$$\overline{\rho\phi''\psi''\chi''} = \overline{\rho\phi\psi\chi}-\overline{\rho\phi\chi}\widetilde{\psi}-\overline{\rho\psi\chi}\widetilde{\phi}+\overline{\rho\chi}\widetilde{\phi}\widetilde{\psi}-\overline{\rho\phi\psi}\widetilde{\chi}+\overline{\rho\phi}\widetilde{\psi}\widetilde{\chi}+\overline{\rho\psi}\widetilde{\phi}\widetilde{\chi}-\overline{\rho}\widetilde{\phi}\widetilde{\psi}\widetilde{\chi} \tag{36}$$

$$\overline{\rho\phi''\psi''\chi''} = \overline{\rho\phi\psi\chi}-\overline{\rho\phi\chi}\widetilde{\psi}-\overline{\rho\psi\chi}\widetilde{\phi}-\overline{\rho\phi\psi}\widetilde{\chi}+\overline{\rho\chi}\widetilde{\phi}\widetilde{\psi}+\overline{\rho\phi}\widetilde{\psi}\widetilde{\chi}+\overline{\rho\psi}\widetilde{\phi}\widetilde{\chi}-\overline{\rho}\widetilde{\phi}\widetilde{\psi}\widetilde{\chi} \tag{37}$$

4 Advection, A

Advection of turbulent kinetic energy:

$$\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \overline{\rho}\widetilde{u}_j\frac{\partial\left[\overline{\rho u_i''u_i''}/(2\overline{\rho})\right]}{\partial x_j} \tag{38}$$

$$\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \frac{\widetilde{u}_j}{2\overline{\rho}}\left[\frac{\partial\overline{\rho u_i''u_i''}}{\partial x_j}\overline{\rho}-\frac{\partial\overline{\rho}}{\partial x_j}\overline{\rho u_i''u_i''}\right] \tag{39}$$

$$\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \frac{\widetilde{u}_j}{2\overline{\rho}}\left[\frac{\partial\overline{\rho u_i''u_i''}}{\partial x_j}\overline{\rho}-\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i''u_i''}\right] \tag{40}$$

Subsisting Eq. 19:

$$\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \frac{\widetilde{u}_j}{2\overline{\rho}}\left[\left(\overline{\frac{\partial\rho}{\partial x_j}u_iu_i}+\overline{\frac{\partial u_i}{\partial x_j}\rho u_i}+\overline{\frac{\partial u_i}{\partial x_j}\rho u_i}-\overline{\frac{\partial\rho}{\partial x_j}}\widetilde{u}_i\widetilde{u}_i-\frac{\partial\widetilde{u}_i}{\partial x_j}\overline{\rho}\widetilde{u}_i-\frac{\partial\widetilde{u}_i}{\partial x_j}\overline{\rho}\widetilde{u}_i\right)\overline{\rho}-\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i''u_i''}\right] \tag{41}$$

$$\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \frac{\widetilde{u}_j}{2\overline{\rho}}\left[\left(\overline{\frac{\partial\rho}{\partial x_j}u_iu_i}+2\overline{\frac{\partial u_i}{\partial x_j}\rho u_i}-\overline{\frac{\partial\rho}{\partial x_j}}\widetilde{u}_i\widetilde{u}_i-2\frac{\partial\widetilde{u}_i}{\partial x_j}\overline{\rho}\widetilde{u}_i\right)\overline{\rho}-\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i''u_i''}\right] \tag{42}$$

Subsisting Eq. 25:

$$\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \frac{\widetilde{u}_j}{2\overline{\rho}}\left[\left(\overline{\frac{\partial\rho}{\partial x_j}u_iu_i}+2\overline{\frac{\partial u_i}{\partial x_j}\rho u_i}-\overline{\frac{\partial\rho}{\partial x_j}}\widetilde{u}_i\widetilde{u}_i-2\left\{\frac{1}{\overline{\rho}}\left(\overline{\frac{\partial\rho}{\partial x_j}u_i}+\overline{\frac{\partial u_i}{\partial x_j}\rho}\right)-\frac{1}{\overline{\rho}^2}\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i}\right\}\overline{\rho}\widetilde{u}_i\right)\overline{\rho}-\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i''u_i''}\right] \tag{43}$$

$$\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \frac{\widetilde{u}_j}{2\overline{\rho}}\left[\left(\overline{\frac{\partial\rho}{\partial x_j}u_iu_i}+2\overline{\frac{\partial u_i}{\partial x_j}\rho u_i}-\overline{\frac{\partial\rho}{\partial x_j}}\widetilde{u}_i\widetilde{u}_i\right)\overline{\rho}-2\left(\overline{\frac{\partial\rho}{\partial x_j}u_i}+\overline{\frac{\partial u_i}{\partial x_j}\rho}-\frac{1}{\overline{\rho}}\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i}\right)\overline{\rho}\widetilde{u}_i-\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i''u_i''}\right] \tag{44}$$

Substituting the definition of a Favre quantity, $\widetilde{\phi}=\frac{\overline{\rho\phi}}{\overline{\rho}}$:

$$\overline{\rho}\widetilde{u}_j\frac{\partial k}{\partial x_j} = \frac{\overline{\rho u_j}}{2\overline{\rho}^2}\left[\left(\overline{\frac{\partial\rho}{\partial x_j}u_iu_i}+2\overline{\frac{\partial u_i}{\partial x_j}\rho u_i}-\overline{\frac{\partial\rho}{\partial x_j}}\frac{\overline{\rho u_i}^2}{\overline{\rho}^2}\right)\overline{\rho}-2\left(\overline{\frac{\partial\rho}{\partial x_j}u_i}+\overline{\frac{\partial u_i}{\partial x_j}\rho}-\frac{1}{\overline{\rho}}\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i}\right)\overline{\rho u_i}-\overline{\frac{\partial\rho}{\partial x_j}}\overline{\rho u_i''u_i''}\right] \tag{45}$$

Terms to collect:

- $\overline{\rho}$

- $\overline{\rho u_i} \rightarrow [\overline{\rho u_1}, \overline{\rho u_2}, \overline{\rho u_3}]$
- $\overline{\rho u_i u_i} \rightarrow \overline{\rho(u_1^2 + u_2^2 + u_3^2)}$
- $\overline{\frac{\partial \rho}{\partial x_j} u_i u_i} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1}(u_1 u_1 + u_2 u_2 + u_3 u_3)}, \overline{\frac{\partial \rho}{\partial x_2}(u_1 u_1 + u_2 u_2 + u_3 u_3)}, \overline{\frac{\partial \rho}{\partial x_3}(u_1 u_1 + u_2 u_2 + u_3 u_3)}\right]$
- $\overline{\frac{\partial u_i}{\partial x_j} \rho u_i} \rightarrow \left[\overline{\left(\frac{\partial u_1}{\partial x_1}\rho u_1 + \frac{\partial u_2}{\partial x_1}\rho u_2 + \frac{\partial u_3}{\partial x_1}\rho u_3\right)}, \overline{\left(\frac{\partial u_1}{\partial x_2}\rho u_1 + \frac{\partial u_2}{\partial x_2}\rho u_2 + \frac{\partial u_3}{\partial x_2}\rho u_3\right)}, \overline{\left(\frac{\partial u_1}{\partial x_3}\rho u_1 + \frac{\partial u_2}{\partial x_3}\rho u_2 + \frac{\partial u_3}{\partial x_3}\rho u_3\right)}\right]$
- $\overline{\frac{\partial \rho}{\partial x_j}} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1}}, \overline{\frac{\partial \rho}{\partial x_2}}, \overline{\frac{\partial \rho}{\partial x_3}}\right]$
- $\overline{\frac{\partial \rho}{\partial x_j} u_i} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1} u_1}, \overline{\frac{\partial \rho}{\partial x_2} u_1}, \overline{\frac{\partial \rho}{\partial x_3} u_1};\ \overline{\frac{\partial \rho}{\partial x_1} u_2}, \overline{\frac{\partial \rho}{\partial x_2} u_2}, \overline{\frac{\partial \rho}{\partial x_3} u_2};\ \overline{\frac{\partial \rho}{\partial x_1} u_3}, \overline{\frac{\partial \rho}{\partial x_2} u_3}, \overline{\frac{\partial \rho}{\partial x_3} u_3}\right]$
- $\overline{\frac{\partial u_i}{\partial x_j} \rho} \rightarrow \left[\overline{\frac{\partial u_1}{\partial x_1}\rho}, \overline{\frac{\partial u_1}{\partial x_2}\rho}, \overline{\frac{\partial u_1}{\partial x_3}\rho};\ \overline{\frac{\partial u_2}{\partial x_1}\rho}, \overline{\frac{\partial u_2}{\partial x_2}\rho}, \overline{\frac{\partial u_2}{\partial x_3}\rho};\ \overline{\frac{\partial u_3}{\partial x_1}\rho}, \overline{\frac{\partial u_3}{\partial x_2}\rho}, \overline{\frac{\partial u_3}{\partial x_3}\rho}\right]$

$\overline{\rho u_i'' u_i''}$ is not explicitly substituted because it can easily be computed from Eq. 31 as:

$$\overline{\rho u_i'' u_i''} = \overline{\rho u_i u_i} - \overline{\rho}\widetilde{u}_i\widetilde{u}_i \tag{46}$$

5 Production, P_k

Production of turbulent kinetic energy:

$$\overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} = -\overline{\rho}\frac{\overline{\rho u_i'' u_j''}}{\overline{\rho}}\frac{\partial \widetilde{u}_i}{\partial x_j} \tag{47}$$

$$\overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} = -\overline{\rho u_i'' u_j''}\frac{\partial \widetilde{u}_i}{\partial x_j} \tag{48}$$

Subsisting Eq. 25:

$$\overline{\rho}\tau_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j} = -\overline{\rho u_i'' u_j''}\left[\frac{1}{\overline{\rho}}\left(\overline{\frac{\partial \rho}{\partial x_j}u_i} + \overline{\frac{\partial u_i}{\partial x_j}\rho}\right) - \frac{1}{\overline{\rho}^2}\overline{\frac{\partial \rho}{\partial x_j}}\,\overline{\rho u_i}\right] \tag{49}$$

$\overline{\rho u_i'' u_j''}$ is not explicitly substituted because it can easily be computed from Eq. 31 as:

$$\overline{\rho u_i'' u_j''} = \overline{\rho u_i u_j} - \overline{\rho}\widetilde{u}_i\widetilde{u}_j \tag{50}$$

Terms to collect:

- $\overline{\rho}$
- $\overline{\rho u_i} \rightarrow [\overline{\rho u_1}, \overline{\rho u_2}, \overline{\rho u_3}]$
- $\overline{\rho u_i u_j} \rightarrow [\overline{\rho u_1 u_1}, \overline{\rho u_1 u_2}, \overline{\rho u_1 u_3};\ \overline{\rho u_2 u_1}, \overline{\rho u_2 u_2}, \overline{\rho u_2 u_3};\ \overline{\rho u_3 u_1}, \overline{\rho u_3 u_2}, \overline{\rho u_3 u_3}]$
- $\overline{\frac{\partial \rho}{\partial x_j}} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1}}, \overline{\frac{\partial \rho}{\partial x_2}}, \overline{\frac{\partial \rho}{\partial x_3}}\right]$
- $\overline{\frac{\partial \rho}{\partial x_j} u_i} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1} u_1}, \overline{\frac{\partial \rho}{\partial x_2} u_1}, \overline{\frac{\partial \rho}{\partial x_3} u_1};\ \overline{\frac{\partial \rho}{\partial x_1} u_2}, \overline{\frac{\partial \rho}{\partial x_2} u_2}, \overline{\frac{\partial \rho}{\partial x_3} u_2};\ \overline{\frac{\partial \rho}{\partial x_1} u_3}, \overline{\frac{\partial \rho}{\partial x_2} u_3}, \overline{\frac{\partial \rho}{\partial x_3} u_3}\right]$
- $\overline{\frac{\partial u_i}{\partial x_j} \rho} \rightarrow \left[\overline{\frac{\partial u_1}{\partial x_1}\rho}, \overline{\frac{\partial u_1}{\partial x_2}\rho}, \overline{\frac{\partial u_1}{\partial x_3}\rho};\ \overline{\frac{\partial u_2}{\partial x_1}\rho}, \overline{\frac{\partial u_2}{\partial x_2}\rho}, \overline{\frac{\partial u_2}{\partial x_3}\rho};\ \overline{\frac{\partial u_3}{\partial x_1}\rho}, \overline{\frac{\partial u_3}{\partial x_2}\rho}, \overline{\frac{\partial u_3}{\partial x_3}\rho}\right]$

6 Viscous Dissipation, $\overline{\rho}\epsilon$

Viscous dissipation of turbulent kinetic energy:

$$\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} = \overline{t_{ij}\frac{\partial (u_i - \widetilde{u}_i)}{\partial x_j}} \tag{51}$$

$$\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} = \overline{t_{ij}\frac{\partial u_i}{\partial x_j}} - \overline{t_{ij}\frac{\partial \widetilde{u}_i}{\partial x_j}} \tag{52}$$

$$\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} = \overline{t_{ij}\frac{\partial u_i}{\partial x_j}} - \overline{t_{ij}}\frac{\partial \widetilde{u}_i}{\partial x_j} \tag{53}$$

Substituting Eq. 25:

$$\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} = \overline{t_{ij}\frac{\partial u_i}{\partial x_j}} - \overline{t_{ij}}\left[\frac{1}{\overline{\rho}}\left(\overline{\frac{\partial \rho}{\partial x_j}u_i} + \overline{\frac{\partial u_i}{\partial x_j}\rho}\right) - \frac{1}{\overline{\rho}^2}\overline{\frac{\partial \rho}{\partial x_j}}\,\overline{\rho u_i}\right] \tag{54}$$

Recalling from Eq. 6 that $t_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}$.

Terms to collect:

- $\overline{\rho}$
- $\overline{\rho u_i} \to [\overline{\rho u_1}, \overline{\rho u_2}, \overline{\rho u_3}]$
- $\overline{t_{ij}} \to [\overline{t_{11}}, \overline{t_{12}}, \overline{t_{13}};\ \overline{t_{21}}, \overline{t_{22}}, \overline{t_{23}};\ \overline{t_{31}}, \overline{t_{32}}, \overline{t_{33}}]$
- $\overline{t_{ij}\frac{\partial u_i}{\partial x_j}} \to \overline{t_{11}\frac{\partial u_1}{\partial x_1} + t_{12}\frac{\partial u_1}{\partial x_2} + t_{13}\frac{\partial u_1}{\partial x_3} + t_{21}\frac{\partial u_2}{\partial x_1} + t_{22}\frac{\partial u_2}{\partial x_2} + t_{23}\frac{\partial u_2}{\partial x_3} + t_{31}\frac{\partial u_3}{\partial x_1} + t_{32}\frac{\partial u_3}{\partial x_2} + t_{33}\frac{\partial u_3}{\partial x_3}}$
- $\overline{\frac{\partial \rho}{\partial x_j}} \to \left[\overline{\frac{\partial \rho}{\partial x_1}}, \overline{\frac{\partial \rho}{\partial x_2}}, \overline{\frac{\partial \rho}{\partial x_3}}\right]$
- $\overline{\frac{\partial \rho}{\partial x_j}u_i} \to \left[\overline{\frac{\partial \rho}{\partial x_1}u_1}, \overline{\frac{\partial \rho}{\partial x_2}u_1}, \overline{\frac{\partial \rho}{\partial x_3}u_1};\ \overline{\frac{\partial \rho}{\partial x_1}u_2}, \overline{\frac{\partial \rho}{\partial x_2}u_2}, \overline{\frac{\partial \rho}{\partial x_3}u_2};\ \overline{\frac{\partial \rho}{\partial x_1}u_3}, \overline{\frac{\partial \rho}{\partial x_2}u_3}, \overline{\frac{\partial \rho}{\partial x_3}u_3}\right]$
- $\overline{\frac{\partial u_i}{\partial x_j}\rho} \to \left[\overline{\frac{\partial u_1}{\partial x_1}\rho}, \overline{\frac{\partial u_1}{\partial x_2}\rho}, \overline{\frac{\partial u_1}{\partial x_3}\rho};\ \overline{\frac{\partial u_2}{\partial x_1}\rho}, \overline{\frac{\partial u_2}{\partial x_2}\rho}, \overline{\frac{\partial u_2}{\partial x_3}\rho};\ \overline{\frac{\partial u_3}{\partial x_1}\rho}, \overline{\frac{\partial u_3}{\partial x_2}\rho}, \overline{\frac{\partial u_3}{\partial x_3}\rho}\right]$

7 Molecular Diffusion, D

Molecular diffusion of turbulent kinetic energy:

$$\frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} = \overline{\frac{\partial u_i''}{\partial x_j}t_{ij}} + \overline{\frac{\partial t_{ij}}{\partial x_j}u_i''} \tag{55}$$

$$\frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} = \underbrace{\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}}}_{\overline{\rho}\epsilon} + \overline{\frac{\partial t_{ij}}{\partial x_j}u_i''} \tag{56}$$

Since the definition of the viscous dissipation arises, there are no additional terms to collect for that component. Continuing with the simplification for u_i'':

$$\frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} = \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} + \overline{\frac{\partial t_{ij}}{\partial x_j}(u_i - \widetilde{u}_i)} \tag{57}$$

$$\frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} = \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} + \overline{\frac{\partial t_{ij}}{\partial x_j}u_i} - \overline{\frac{\partial t_{ij}}{\partial x_j}\widetilde{u}_i} \tag{58}$$

$$\frac{\partial \overline{u_i'' t_{ij}}}{\partial x_j} = \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} + \overline{\frac{\partial t_{ij}}{\partial x_j}u_i} - \frac{\partial \overline{t_{ij}}}{\partial x_j}\widetilde{u}_i \tag{59}$$

The spatial derivatives of the viscous stress tensor present challenges because they result in second derivatives of the velocity components. Therefore, they are left alone, and the spatial derivatives are to be computed after-the-fact on the mean quantity. In practice, the boundary layer assumptions may be invoked such that $\frac{\partial\overline{(\cdot)}}{\partial x_1}, \frac{\partial\overline{(\cdot)}}{\partial x_3} \ll \frac{\partial\overline{(\cdot)}}{\partial x_2}$ and $\overline{u_2}, \overline{u_3} \ll \overline{u_1}$. Subsequently the task for simplification of Eq. 59 is to bring the product of the velocity with the divergence of the viscous stress tensor to be inside the derivative to have the divergence of the product of the velocity and viscous stress tensor.

$$\frac{\partial\overline{u_i'' t_{ij}}}{\partial x_j} = \overline{t_{ij}\frac{\partial u_i''}{\partial x_j}} + \frac{\overline{\partial(t_{ij}u_i)}}{\partial x_j} - \overline{\frac{\partial u_i}{\partial x_j}t_{ij}} - \frac{\partial\overline{t_{ij}}}{\partial x_j}\widetilde{u_i} \tag{60}$$

$$\frac{\partial\overline{u_i'' t_{ij}}}{\partial x_j} = \underbrace{\overline{t_{ij}\frac{\partial u_i''}{\partial x_j}}}_{\overline{\rho}\epsilon} + \frac{\partial\overline{(t_{ij}u_i)}}{\partial x_j} - \overline{t_{ij}\frac{\partial u_i}{\partial x_j}} - \frac{\partial\overline{t_{ij}}}{\partial x_j}\widetilde{u_i} \tag{61}$$

Many of these terms to collect were previously seen in the viscous dissipation.

Terms to collect:

- $\overline{\rho}$
- $\overline{\rho u_i} \to [\overline{\rho u_1}, \overline{\rho u_2}, \overline{\rho u_3}]$
- $\overline{t_{ij}} \to [\overline{t_{11}}, \overline{t_{12}}, \overline{t_{13}};\ \overline{t_{21}}, \overline{t_{22}}, \overline{t_{23}};\ \overline{t_{31}}, \overline{t_{32}}, \overline{t_{33}}]$
- $\overline{t_{ij}\frac{\partial u_i}{\partial x_j}} \to \overline{t_{11}\frac{\partial u_1}{\partial x_1} + t_{12}\frac{\partial u_1}{\partial x_2} + t_{13}\frac{\partial u_1}{\partial x_3} + t_{21}\frac{\partial u_2}{\partial x_1} + t_{22}\frac{\partial u_2}{\partial x_2} + t_{23}\frac{\partial u_2}{\partial x_3} + t_{31}\frac{\partial u_3}{\partial x_1} + t_{32}\frac{\partial u_3}{\partial x_2} + t_{33}\frac{\partial u_3}{\partial x_3}}$
- $\overline{\frac{\partial \rho}{\partial x_j}} \to \left[\overline{\frac{\partial \rho}{\partial x_1}}, \overline{\frac{\partial \rho}{\partial x_2}}, \overline{\frac{\partial \rho}{\partial x_3}}\right]$
- $\overline{\frac{\partial \rho}{\partial x_j}u_i} \to \left[\overline{\frac{\partial \rho}{\partial x_1}u_1}, \overline{\frac{\partial \rho}{\partial x_2}u_1}, \overline{\frac{\partial \rho}{\partial x_3}u_1};\ \overline{\frac{\partial \rho}{\partial x_1}u_2}, \overline{\frac{\partial \rho}{\partial x_2}u_2}, \overline{\frac{\partial \rho}{\partial x_3}u_2};\ \overline{\frac{\partial \rho}{\partial x_1}u_3}, \overline{\frac{\partial \rho}{\partial x_2}u_3}, \overline{\frac{\partial \rho}{\partial x_3}u_3}\right]$
- $\overline{\frac{\partial u_i}{\partial x_j}\rho} \to \left[\overline{\frac{\partial u_1}{\partial x_1}\rho}, \overline{\frac{\partial u_1}{\partial x_2}\rho}, \overline{\frac{\partial u_1}{\partial x_3}\rho};\ \overline{\frac{\partial u_2}{\partial x_1}\rho}, \overline{\frac{\partial u_2}{\partial x_2}\rho}, \overline{\frac{\partial u_2}{\partial x_3}\rho};\ \overline{\frac{\partial u_3}{\partial x_1}\rho}, \overline{\frac{\partial u_3}{\partial x_2}\rho}, \overline{\frac{\partial u_3}{\partial x_3}\rho}\right]$
- $\overline{t_{ij}u_i} \to \left[\overline{(t_{11}u_1 + t_{21}u_2 + t_{31}u_3)}, \overline{(t_{12}u_1 + t_{22}u_2 + t_{32}u_3)}, \overline{(t_{13}u_1 + t_{23}u_2 + t_{33}u_3)}\right]$

8 Pressure Diffusion, Π_t

Pressure diffusion of turbulent kinetic energy:

$$\frac{\partial\overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial u_j''}{\partial x_j}P'} + \overline{\frac{\partial P'}{\partial x_j}u_j''} \tag{62}$$

$$\frac{\partial\overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial(u_j - \widetilde{u_j})}{\partial x_j}(P - \overline{P})} + \overline{\frac{\partial(P - \overline{P})}{\partial x_j}(u_j - \widetilde{u_j})} \tag{63}$$

$$\frac{\partial\overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial u_j}{\partial x_j}P} - \overline{\frac{\partial u_j}{\partial x_j}\overline{P}} - \overline{\frac{\partial \widetilde{u_j}}{\partial x_j}P} + \overline{\frac{\partial \widetilde{u_j}}{\partial x_j}\overline{P}} + \overline{\frac{\partial P}{\partial x_j}u_j} - \overline{\frac{\partial \overline{P}}{\partial x_j}u_j} - \overline{\frac{\partial P}{\partial x_j}\widetilde{u_j}} + \overline{\frac{\partial \overline{P}}{\partial x_j}\widetilde{u_j}} \tag{64}$$

$$\frac{\partial\overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial u_j}{\partial x_j}P} - \frac{\partial \overline{u_j}}{\partial x_j}\overline{P} - \cancel{\frac{\partial \widetilde{u_j}}{\partial x_j}\overline{P}} + \cancel{\frac{\partial \widetilde{u_j}}{\partial x_j}\overline{P}} + \overline{\frac{\partial P}{\partial x_j}u_j} - \frac{\partial \overline{P}}{\partial x_j}\overline{u_j} - \cancel{\frac{\partial \overline{P}}{\partial x_j}\widetilde{u_j}} + \cancel{\frac{\partial \overline{P}}{\partial x_j}\widetilde{u_j}} \tag{65}$$

$$\frac{\partial\overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial u_j}{\partial x_j}P} - \overline{\frac{\partial u_j}{\partial x_j}}\,\overline{P} + \overline{\frac{\partial P}{\partial x_j}u_j} - \overline{\frac{\partial P}{\partial x_j}}\,\overline{u_j} \tag{66}$$

For a perfect gas, pressure P can be replaced with temperature T and density ρ via the equation of state $P = \rho RT$, where R is the specific gas constant. This substitution is made for the spatial derivatives of pressure only.

$$\frac{\partial \overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial u_j}{\partial x_j} P} - \overline{\frac{\partial u_j}{\partial x_j}}\,\overline{P} + \overline{\frac{\partial (\rho RT)}{\partial x_j} u_j} - \overline{\frac{\partial (\rho RT)}{\partial x_j}}\,\overline{u_j} \tag{67}$$

$$\frac{\partial \overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial u_j}{\partial x_j} P} - \overline{\frac{\partial u_j}{\partial x_j}}\,\overline{P} + R\overline{\frac{\partial (\rho T)}{\partial x_j} u_j} - R\overline{\frac{\partial (\rho T)}{\partial x_j}}\,\overline{u_j} \tag{68}$$

$$\frac{\partial \overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial u_j}{\partial x_j} P} - \overline{\frac{\partial u_j}{\partial x_j}}\,\overline{P} + R\overline{\left(\frac{\partial \rho}{\partial x_j} T + \frac{\partial T}{\partial x_j}\rho\right) u_j} - R\overline{\left(\frac{\partial \rho}{\partial x_j} T + \frac{\partial T}{\partial x_j}\rho\right)}\,\overline{u_j} \tag{69}$$

$$\frac{\partial \overline{u_j'' P'}}{\partial x_j} = \overline{\frac{\partial u_j}{\partial x_j} P} - \overline{\frac{\partial u_j}{\partial x_j}}\,\overline{P} + R\overline{\frac{\partial \rho}{\partial x_j} T u_j} + R\overline{\frac{\partial T}{\partial x_j}\rho u_j} - R\overline{\frac{\partial \rho}{\partial x_j} T}\,\overline{u_j} - R\overline{\frac{\partial T}{\partial x_j}\rho}\,\overline{u_j} \tag{70}$$

Terms to collect:

- $\overline{P}$
- $\overline{u_j} \rightarrow [\overline{u_1}, \overline{u_2}, \overline{u_3}]$
- $\overline{\frac{\partial u_k}{\partial x_k}} \rightarrow \overline{\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}}$
- $\overline{P\frac{\partial u_k}{\partial x_k}} \rightarrow \overline{P\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}$
- $\overline{\frac{\partial \rho}{\partial x_k} T u_k} \rightarrow \overline{T\left(\frac{\partial \rho}{\partial x_1}u_1 + \frac{\partial \rho}{\partial x_2}u_2 + \frac{\partial \rho}{\partial x_3}u_3\right)}$
- $\overline{\frac{\partial T}{\partial x_k} \rho u_k} \rightarrow \overline{\rho\left(\frac{\partial T}{\partial x_1}u_1 + \frac{\partial T}{\partial x_2}u_2 + \frac{\partial T}{\partial x_3}u_3\right)}$
- $\overline{\frac{\partial \rho}{\partial x_j} T} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1} T}, \overline{\frac{\partial \rho}{\partial x_2} T}, \overline{\frac{\partial \rho}{\partial x_3} T}\right]$
- $\overline{\frac{\partial T}{\partial x_j} \rho} \rightarrow \left[\overline{\frac{\partial T}{\partial x_1} \rho}, \overline{\frac{\partial T}{\partial x_2} \rho}, \overline{\frac{\partial T}{\partial x_3} \rho}\right]$

9 Turbulent Transport, T_k

Turbulent transport of turbulent kinetic energy:

$$\frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_j}\left[\overline{\rho(u_i - \widetilde{u_i})(u_i - \widetilde{u_i})(u_j - \widetilde{u_j})}\right] \tag{71}$$

$$\frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_j}\left[\overline{\rho(u_i u_i u_j - u_i u_i \widetilde{u_j} - 2u_i\widetilde{u_i}u_j + 2u_i\widetilde{u_i}\widetilde{u_j} + \widetilde{u_i}\widetilde{u_i}u_j - \widetilde{u_i}\widetilde{u_i}\widetilde{u_j})}\right] \tag{72}$$

$$\frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_j}\left[\overline{\rho u_i u_i u_j} - \overline{\rho u_i u_i}\widetilde{u_j} - 2\overline{\rho u_i u_j}\widetilde{u_i} + 2\overline{\rho u_i}\widetilde{u_i}\widetilde{u_j} + \overline{\rho u_j}\widetilde{u_i}\widetilde{u_i} - \overline{\rho}\widetilde{u_i}\widetilde{u_i}\widetilde{u_j})\right] \tag{73}$$

$$\frac{1}{2}\frac{\partial \overline{\rho u_i'' u_i'' u_j''}}{\partial x_j} = \frac{1}{2}\left(\underbrace{\frac{\partial\left[\overline{\rho u_i u_i u_j}\right]}{\partial x_j}}_{T1} - \underbrace{\frac{\partial\left[\overline{\rho u_i u_i}\widetilde{u_j}\right]}{\partial x_j}}_{T2} - 2\underbrace{\frac{\partial\left[\overline{\rho u_i u_j}\widetilde{u_i}\right]}{\partial x_j}}_{T3} + 2\underbrace{\frac{\partial\left[\overline{\rho u_i}\widetilde{u_i}\widetilde{u_j}\right]}{\partial x_j}}_{T4} + \underbrace{\frac{\partial\left[\overline{\rho u_j}\widetilde{u_i}\widetilde{u_i}\right]}{\partial x_j}}_{T5} - \underbrace{\frac{\partial\left[\overline{\rho}\widetilde{u_i}\widetilde{u_i}\widetilde{u_j}\right]}{\partial x_j}}_{T6}\right) \tag{74}$$

For clarity, each sub-term will be expanded individually.

T1:

$$\frac{\partial\left[\overline{\rho u_i u_i u_j}\right]}{\partial x_j} = \overline{\frac{\partial\left[\rho u_i u_i u_j\right]}{\partial x_j}} \tag{75}$$

$$\frac{\partial\left[\overline{\rho u_i u_i u_j}\right]}{\partial x_j} = \overline{\frac{\partial \rho}{\partial x_j} u_i u_i u_j + 2\frac{\partial u_i}{\partial x_j}\rho u_i u_j + \frac{\partial u_j}{\partial x_j}\rho u_i u_i} \tag{76}$$

T2:

$$\frac{\partial\left[\overline{\rho u_i u_i}\widetilde{u_j}\right]}{\partial x_j} = \frac{\partial\left[\overline{\rho u_i u_i}\right]}{\partial x_j}\widetilde{u_j} + \frac{\partial \widetilde{u_j}}{\partial x_j}\overline{\rho u_i u_i} \tag{77}$$

$$\frac{\partial\left[\overline{\rho u_i u_i}\widetilde{u_j}\right]}{\partial x_j} = \overline{\frac{\partial\left[\rho u_i u_i\right]}{\partial x_j}}\widetilde{u_j} + \frac{\partial \widetilde{u_j}}{\partial x_j}\overline{\rho u_i u_i} \tag{78}$$

$$\frac{\partial\left[\overline{\rho u_i u_i}\widetilde{u_j}\right]}{\partial x_j} = \overline{\left(\frac{\partial \rho}{\partial x_j} u_i u_i + 2\frac{\partial u_i}{\partial x_j}\rho u_i\right)}\widetilde{u_j} + \frac{\partial \widetilde{u_j}}{\partial x_j}\overline{\rho u_i u_i} \tag{79}$$

T3:

$$\frac{\partial\left[\overline{\rho u_i u_j}\widetilde{u_i}\right]}{\partial x_j} = \frac{\partial\left[\overline{\rho u_i u_j}\right]}{\partial x_j}\widetilde{u_i} + \frac{\partial \widetilde{u_i}}{\partial x_j}\overline{\rho u_i u_j} \tag{80}$$

$$\frac{\partial\left[\overline{\rho u_i u_j}\widetilde{u_i}\right]}{\partial x_j} = \overline{\frac{\partial\left[\rho u_i u_j\right]}{\partial x_j}}\widetilde{u_i} + \frac{\partial \widetilde{u_i}}{\partial x_j}\overline{\rho u_i u_j} \tag{81}$$

$$\frac{\partial\left[\overline{\rho u_i u_j}\widetilde{u_i}\right]}{\partial x_j} = \overline{\left(\frac{\partial \rho}{\partial x_j} u_i u_j + \frac{\partial u_i}{\partial x_j}\rho u_j + \frac{\partial u_j}{\partial x_j}\rho u_i\right)}\widetilde{u_i} + \frac{\partial \widetilde{u_i}}{\partial x_j}\overline{\rho u_i u_j} \tag{82}$$

T4:

$$\frac{\partial\left[\overline{\rho u_i}\widetilde{u_i}\widetilde{u_j}\right]}{\partial x_j} = \frac{\partial\left[\overline{\rho u_i}\right]}{\partial x_j}\widetilde{u_i}\widetilde{u_j} + \frac{\partial\left[\widetilde{u_i}\widetilde{u_j}\right]}{\partial x_j}\overline{\rho u_i} \tag{83}$$

$$\frac{\partial\left[\overline{\rho u_i}\widetilde{u_i}\widetilde{u_j}\right]}{\partial x_j} = \overline{\frac{\partial\left[\rho u_i\right]}{\partial x_j}}\widetilde{u_i}\widetilde{u_j} + \frac{\partial\left[\widetilde{u_i}\widetilde{u_j}\right]}{\partial x_j}\overline{\rho u_i} \tag{84}$$

$$\frac{\partial\left[\overline{\rho u_i}\widetilde{u_i}\widetilde{u_j}\right]}{\partial x_j} = \overline{\left(\frac{\partial \rho}{\partial x_j} u_i\right)}\widetilde{u_i}\widetilde{u_j} + \overline{\left(\frac{\partial u_i}{\partial x_j}\rho\right)}\widetilde{u_i}\widetilde{u_j} + \frac{\partial \widetilde{u_j}}{\partial x_j}\widetilde{u_i}\overline{\rho u_i} + \frac{\partial \widetilde{u_i}}{\partial x_j}\widetilde{u_j}\overline{\rho u_i} \tag{85}$$

$$\tag{86}$$

T5:

$$\frac{\partial\left[\overline{\rho u_j}\widetilde{u_i}\widetilde{u_i}\right]}{\partial x_j} = \frac{\partial\left[\overline{\rho u_j}\right]}{\partial x_j}\widetilde{u_i}\widetilde{u_i} + \frac{\partial\left[\widetilde{u_i}\widetilde{u_i}\right]}{\partial x_j}\overline{\rho u_j} \tag{87}$$

$$\frac{\partial\left[\overline{\rho u_j}\widetilde{u_i}\widetilde{u_i}\right]}{\partial x_j} = \overline{\frac{\partial\left[\rho u_j\right]}{\partial x_j}}\widetilde{u_i}\widetilde{u_i} + \frac{\partial\left[\widetilde{u_i}\widetilde{u_i}\right]}{\partial x_j}\overline{\rho u_j} \tag{88}$$

$$\frac{\partial\left[\overline{\rho u_j}\widetilde{u_i}\widetilde{u_i}\right]}{\partial x_j} = \overline{\left(\frac{\partial \rho}{\partial x_j} u_j\right)}\widetilde{u_i}\widetilde{u_i} + \overline{\left(\frac{\partial u_j}{\partial x_j}\rho\right)}\widetilde{u_i}\widetilde{u_i} + 2\frac{\partial \widetilde{u_i}}{\partial x_j}\widetilde{u_i}\overline{\rho u_j} \tag{89}$$

$$\tag{90}$$

T6:

$$\frac{\partial\left[\overline{\rho}\widetilde{u_i}\widetilde{u_i}\widetilde{u_j}\right]}{\partial x_j} = \overline{\frac{\partial \rho}{\partial x_j}}\widetilde{u_i}\widetilde{u_i}\widetilde{u_j} + 2\frac{\partial \widetilde{u_i}}{\partial x_j}\overline{\rho}\widetilde{u_i}\widetilde{u_j} + \frac{\partial \widetilde{u_j}}{\partial x_j}\overline{\rho}\widetilde{u_i}\widetilde{u_i} \tag{91}$$

From Eq. 25:

$$\frac{\partial \widetilde{u_i}}{\partial x_j} = \frac{1}{\overline{\rho}}\left(\overline{\frac{\partial \rho}{\partial x_j}u_i} + \overline{\frac{\partial u_i}{\partial x_j}\rho}\right) - \frac{1}{\overline{\rho}^2}\overline{\frac{\partial \rho}{\partial x_j}}\,\overline{\rho u_i} \tag{92}$$

$$\text{and} \tag{93}$$

$$\frac{\partial \widetilde{u_j}}{\partial x_j} = \frac{1}{\overline{\rho}}\left(\overline{\frac{\partial \rho}{\partial x_j}u_j} + \overline{\frac{\partial u_j}{\partial x_j}\rho}\right) - \frac{1}{\overline{\rho}^2}\overline{\frac{\partial \rho}{\partial x_j}}\,\overline{\rho u_j} \tag{94}$$

Terms to collect:

- $\overline{\rho}$
- $\overline{\rho u_i} \rightarrow [\overline{\rho u_1}, \overline{\rho u_3}, \overline{\rho u_3}]$
- $\overline{\rho u_i u_j} \rightarrow [\overline{\rho u_1 u_1}, \overline{\rho u_1 u_2}, \overline{\rho u_1 u_3};\ \overline{\rho u_2 u_1}, \overline{\rho u_2 u_2}, \overline{\rho u_2 u_3};\ \overline{\rho u_3 u_1}, \overline{\rho u_3 u_2}, \overline{\rho u_3 u_3}]$
- $\overline{\frac{\partial \rho}{\partial x_j}u_i u_i u_j + 2\frac{\partial u_i}{\partial x_j}\rho u_i u_j + \frac{\partial u_j}{\partial x_j}\rho u_i u_i} \rightarrow$

$$\overline{\frac{\partial \rho}{\partial x_1}(u_1u_1 + u_2u_2 + u_3u_3)u_1 + \frac{\partial \rho}{\partial x_2}(u_1u_1 + u_2u_2 + u_3u_3)u_2 + \frac{\partial \rho}{\partial x_3}(u_1u_1 + u_2u_2 + u_3u_3)u_3 +}$$
$$\overline{2\left[\left(\frac{\partial u_1}{\partial x_1}\rho u_1 + \frac{\partial u_2}{\partial x_1}\rho u_2 + \frac{\partial u_3}{\partial x_1}\rho u_3\right)u_1 + \left(\frac{\partial u_1}{\partial x_2}\rho u_1 + \frac{\partial u_2}{\partial x_2}\rho u_2 + \frac{\partial u_3}{\partial x_2}\rho u_3\right)u_2 + \left(\frac{\partial u_1}{\partial x_3}\rho u_1 + \frac{\partial u_2}{\partial x_3}\rho u_2 + \frac{\partial u_3}{\partial x_3}\rho u_3\right)u_3\right]}$$
$$\overline{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\rho\,(u_1u_1 + u_2u_2 + u_3u_3)}$$

- $\overline{\frac{\partial \rho}{\partial x_j}u_i u_i} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1}(u_1u_1 + u_2u_2 + u_3u_3)}, \overline{\frac{\partial \rho}{\partial x_2}(u_1u_1 + u_2u_2 + u_3u_3)}, \overline{\frac{\partial \rho}{\partial x_3}(u_1u_1 + u_2u_2 + u_3u_3)}\right]$
- $\overline{\frac{\partial \rho}{\partial x_j}u_i u_j} \rightarrow \left[\overline{\left(\frac{\partial \rho}{\partial x_1}u_1 + \frac{\partial \rho}{\partial x_2}u_2 + \frac{\partial \rho}{\partial x_3}u_3\right)u_1}, \overline{\left(\frac{\partial \rho}{\partial x_1}u_1 + \frac{\partial \rho}{\partial x_2}u_2 + \frac{\partial \rho}{\partial x_3}u_3\right)u_2}, \overline{\left(\frac{\partial \rho}{\partial x_1}u_1 + \frac{\partial \rho}{\partial x_2}u_2 + \frac{\partial \rho}{\partial x_3}u_3\right)u_3}\right]$
- $\overline{\frac{\partial u_i}{\partial x_j}\rho u_i} \rightarrow \left[\overline{\left(\frac{\partial u_1}{\partial x_1}\rho u_1 + \frac{\partial u_2}{\partial x_1}\rho u_2 + \frac{\partial u_3}{\partial x_1}\rho u_3\right)}, \overline{\left(\frac{\partial u_1}{\partial x_2}\rho u_1 + \frac{\partial u_2}{\partial x_2}\rho u_2 + \frac{\partial u_3}{\partial x_2}\rho u_3\right)}, \overline{\left(\frac{\partial u_1}{\partial x_3}\rho u_1 + \frac{\partial u_2}{\partial x_3}\rho u_2 + \frac{\partial u_3}{\partial x_3}\rho u_3\right)}\right]$
- $\overline{\frac{\partial u_i}{\partial x_j}\rho u_j} \rightarrow \left[\overline{\left(\frac{\partial u_1}{\partial x_1}\rho u_1 + \frac{\partial u_1}{\partial x_2}\rho u_2 + \frac{\partial u_1}{\partial x_3}\rho u_3\right)}, \overline{\left(\frac{\partial u_2}{\partial x_1}\rho u_1 + \frac{\partial u_2}{\partial x_2}\rho u_2 + \frac{\partial u_2}{\partial x_3}\rho u_3\right)}, \overline{\left(\frac{\partial u_3}{\partial x_1}\rho u_1 + \frac{\partial u_3}{\partial x_2}\rho u_2 + \frac{\partial u_3}{\partial x_3}\rho u_3\right)}\right]$
- $\overline{\frac{\partial u_j}{\partial x_j}\rho u_i} \rightarrow \left[\overline{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\rho u_1}, \overline{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\rho u_2}, \overline{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\rho u_3}\right]$
- $\overline{\frac{\partial \rho}{\partial x_j}} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1}}, \overline{\frac{\partial \rho}{\partial x_2}}, \overline{\frac{\partial \rho}{\partial x_3}}\right]$
- $\overline{\frac{\partial \rho}{\partial x_j}u_i} \rightarrow \left[\overline{\frac{\partial \rho}{\partial x_1}u_1}, \overline{\frac{\partial \rho}{\partial x_2}u_1}, \overline{\frac{\partial \rho}{\partial x_3}u_1};\ \overline{\frac{\partial \rho}{\partial x_1}u_2}, \overline{\frac{\partial \rho}{\partial x_2}u_2}, \overline{\frac{\partial \rho}{\partial x_3}u_2};\ \overline{\frac{\partial \rho}{\partial x_1}u_3}, \overline{\frac{\partial \rho}{\partial x_2}u_3}, \overline{\frac{\partial \rho}{\partial x_3}u_3}\right]$
- $\overline{\frac{\partial u_i}{\partial x_j}\rho} \rightarrow \left[\overline{\frac{\partial u_1}{\partial x_1}\rho}, \overline{\frac{\partial u_1}{\partial x_2}\rho}, \overline{\frac{\partial u_1}{\partial x_3}\rho};\ \overline{\frac{\partial u_2}{\partial x_1}\rho}, \overline{\frac{\partial u_2}{\partial x_2}\rho}, \overline{\frac{\partial u_2}{\partial x_3}\rho};\ \overline{\frac{\partial u_3}{\partial x_1}\rho}, \overline{\frac{\partial u_3}{\partial x_2}\rho}, \overline{\frac{\partial u_3}{\partial x_3}\rho}\right]$
- $\overline{\rho u_j u_i u_i} \rightarrow \left[\overline{\rho u_1(u_1u_1 + u_2u_2 + u_3u_3)}, \overline{\rho u_2(u_1u_1 + u_2u_2 + u_3u_3)}, \overline{\rho u_3(u_1u_1 + u_2u_2 + u_3u_3)}\right]$

10 Pressure Work, Π_w

Pressure work on turbulent kinetic energy:

$$\overline{u_i''}\frac{\partial\overline{P}}{\partial x_i} = \overline{(u_i - \widetilde{u}_i)}\overline{\frac{\partial P}{\partial x_i}} \tag{95}$$

$$\overline{u_i''}\frac{\partial\overline{P}}{\partial x_i} = (\overline{u_i} - \widetilde{u}_i)\overline{\frac{\partial P}{\partial x_i}} \tag{96}$$

$$\overline{u_i''}\frac{\partial\overline{P}}{\partial x_i} = \overline{u_i}\overline{\frac{\partial P}{\partial x_i}} - \widetilde{u}_i\overline{\frac{\partial P}{\partial x_i}} \tag{97}$$

Substitute pressure via the equation of state:

$$\overline{u_i''}\frac{\partial\overline{P}}{\partial x_i} = \overline{u_i}\overline{\frac{\partial(\rho RT)}{\partial x_i}} - \widetilde{u}_i\overline{\frac{\partial(\rho RT)}{\partial x_i}} \tag{98}$$

$$\overline{u_i''}\frac{\partial\overline{P}}{\partial x_i} = \overline{u_i}R\overline{\frac{\partial(\rho T)}{\partial x_i}} - \widetilde{u}_iR\overline{\frac{\partial(\rho T)}{\partial x_i}} \tag{99}$$

$$\overline{u_i''}\frac{\partial\overline{P}}{\partial x_i} = \overline{u_i}R\overline{\frac{\partial\rho}{\partial x_i}T} + \overline{u_i}R\overline{\frac{\partial T}{\partial x_i}\rho} - \widetilde{u}_iR\overline{\frac{\partial\rho}{\partial x_i}T} - \widetilde{u}_iR\overline{\frac{\partial T}{\partial x_i}\rho} \tag{100}$$

Terms to collect:

- $\overline{\rho}$
- $\overline{\rho u_i} \to [\overline{\rho u_1}, \overline{\rho u_2}, \overline{\rho u_3}]$
- $\overline{u_i} \to [\overline{u_1}, \overline{u_2}, \overline{u_3}]$
- $\overline{\frac{\partial\rho}{\partial x_i}T} \to \left[\overline{\frac{\partial\rho}{\partial x_1}T}, \overline{\frac{\partial\rho}{\partial x_2}T}, \overline{\frac{\partial\rho}{\partial x_3}T}\right]$
- $\overline{\frac{\partial T}{\partial x_i}\rho} \to \left[\overline{\frac{\partial T}{\partial x_1}\rho}, \overline{\frac{\partial T}{\partial x_2}\rho}, \overline{\frac{\partial T}{\partial x_3}\rho}\right]$

11 Pressure Dilatation, Π_d

Pressure dilatation of turbulent kinetic energy:

$$\overline{P'\frac{\partial u_i''}{\partial x_i}} = \overline{(P - \overline{P})\frac{\partial(u_i - \widetilde{u}_i)}{\partial x_i}} \tag{101}$$

$$\overline{P'\frac{\partial u_i''}{\partial x_i}} = \overline{P\frac{\partial u_i}{\partial x_i}} - \overline{P}\overline{\frac{\partial u_i}{\partial x_i}} - \overline{P\frac{\partial\widetilde{u}_i}{\partial x_i}} + \overline{P}\overline{\frac{\partial\widetilde{u}_i}{\partial x_i}} \tag{102}$$

$$\overline{P'\frac{\partial u_i''}{\partial x_i}} = \overline{P\frac{\partial u_i}{\partial x_i}} - \overline{P}\overline{\frac{\partial u_i}{\partial x_i}} - \cancel{\overline{P}\frac{\partial\widetilde{u}_i}{\partial x_i}} + \cancel{\overline{P}\frac{\partial\widetilde{u}_i}{\partial x_i}} \tag{103}$$

$$\overline{P'\frac{\partial u_i''}{\partial x_i}} = \overline{P\frac{\partial u_i}{\partial x_i}} - \overline{P}\overline{\frac{\partial u_i}{\partial x_i}} \tag{104}$$

Terms to collect:

- $\overline{P}$
- $\overline{P\frac{\partial u_k}{\partial x_k}} \to \overline{P\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}$
- $\overline{\frac{\partial u_k}{\partial x_k}} \to \overline{\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}}$

12 Heat Transfer

For heat transfer, the two quantities of interest presently are the molecular heat flux ($q_{Li} = \overline{q_i}$) and the turbulent heat flux (q_{Ti}). To start, the molecular heat flux is as follows, where $q_i = \overline{q_i} + q_i'$:

$$q_i = -\kappa \frac{\partial T}{\partial x_i} \rightarrow q_{Li} = \overline{q_i} = -\overline{\kappa \frac{\partial T}{\partial x_i}} \tag{105}$$

$$\overline{q_i} = -\overline{(\overline{\kappa} + \kappa') \frac{\partial(\overline{T} + T')}{\partial x_i}} \tag{106}$$

$$\overline{q_i} = -\overline{\overline{\kappa} \frac{\partial \overline{T}}{\partial x_i}} - \overline{\overline{\kappa} \frac{\partial T'}{\partial x_i}} - \overline{\kappa' \frac{\partial \overline{T}}{\partial x_i}} - \overline{\kappa' \frac{\partial T'}{\partial x_i}} \tag{107}$$

$$\overline{q_i} = -\overline{\kappa} \frac{\partial \overline{T}}{\partial x_i} - \overline{\kappa} \frac{\partial \cancelto{0}{\overline{T'}}}{\partial x_i} - \cancelto{0}{\overline{\kappa'}} \frac{\partial \overline{T}}{\partial x_i} - \overline{\kappa' \frac{\partial T'}{\partial x_i}} \tag{108}$$

$$\overline{q_i} = -\overline{\kappa} \frac{\partial \overline{T}}{\partial x_i} - \overline{\kappa' \frac{\partial T'}{\partial x_i}} \tag{109}$$

Often the thermal conductivity - temperature gradient fluctuation correlation $\overline{\kappa' \frac{\partial T'}{\partial x_i}}$ is neglected. The turbulent heat transfer is given as follows, making use of constant specific heats to write the enthalpy h in terms of temperature T:

$$q_{Ti} = \overline{\rho u_i'' h''} \tag{110}$$

$$q_{Ti} = c_p \overline{\rho u_i'' T''} \tag{111}$$

The fluctuation correlation can be re-written similarly to Eq. 31:

$$c_p \overline{\rho u_i'' T''} = c_p \overline{\rho u_i T} - c_p \overline{\rho} \widetilde{u}_i \widetilde{T} \tag{112}$$

$$c_p \overline{\rho u_i'' T''} = c_p \overline{\rho u_i T} - c_p \overline{\rho} \frac{\overline{\rho u_i}}{\overline{\rho}} \frac{\overline{\rho T}}{\overline{\rho}} \tag{113}$$

$$c_p \overline{\rho u_i'' T''} = c_p \overline{\rho u_i T} - c_p \overline{\rho u_i} \frac{\overline{\rho T}}{\overline{\rho}} \tag{114}$$

Terms to collect:

- $\overline{\rho}$
- $\overline{\kappa}$
- $\overline{T}$
- $\overline{\kappa \frac{\partial T}{\partial x_i}} \rightarrow \left[\overline{\kappa \frac{\partial T}{\partial x_1}}, \overline{\kappa \frac{\partial T}{\partial x_2}}, \overline{\kappa \frac{\partial T}{\partial x_3}}\right]$
- $\overline{\rho T}$
- $\overline{\rho u_i} \rightarrow [\overline{\rho u_1}, \overline{\rho u_2}, \overline{\rho u_3}]$
- $\overline{\rho u_i T} \rightarrow [\overline{\rho u_1 T}, \overline{\rho u_2 T}, \overline{\rho u_3 T}]$
- $\overline{\rho T T}$ (for T_{RMS})

13 List of Unique Terms to Output

Many of the same outputs arose in the individual budget terms. Additionally, some terms are redundant due to symmetry (i.e. viscous stress tensor). Therefore, the list of unique terms to output is as follows:

1. $stats\_time$
2. x_1
3. x_2
4. x_3
5. $\overline{\rho}$
6. $\overline{u_1}$
7. $\overline{u_2}$
8. $\overline{u_3}$
9. $\overline{T}$
10. $\overline{P}$
11. $\overline{\mu}$
12. $\overline{\kappa}$
13. $\overline{c_p}$
14. $\overline{c_v}$
15. $\overline{\rho u_1}$
16. $\overline{\rho u_2}$
17. $\overline{\rho u_3}$
18. $\overline{\rho T}$
19. $\overline{\rho u_1 u_1}$
20. $\overline{\rho u_1 u_2}$
21. $\overline{\rho u_1 u_3}$
22. $\overline{\rho u_2 u_2}$
23. $\overline{\rho u_2 u_3}$
24. $\overline{\rho u_3 u_3}$
25. $\overline{\rho u_1 T}$
26. $\overline{\rho u_2 T}$
27. $\overline{\rho u_3 T}$
28. $\overline{\rho T T}$
29. $\overline{\rho u_1 (u_1 u_1 + u_2 u_2 + u_3 u_3)}$

30. $\overline{\rho u_2(u_1u_1+u_2u_2+u_3u_3)}$
31. $\overline{\rho u_3(u_1u_1+u_2u_2+u_3u_3)}$
32. $\overline{t_{11}}$
33. $\overline{t_{12}}$
34. $\overline{t_{13}}$
35. $\overline{t_{22}}$
36. $\overline{t_{23}}$
37. $\overline{t_{33}}$
38. $\overline{t_{11}u_1+t_{12}u_2+t_{13}u_3}$
39. $\overline{t_{12}u_1+t_{22}u_2+t_{23}u_3}$
40. $\overline{t_{13}u_1+t_{23}u_2+t_{33}u_3}$
41. $\overline{t_{11}\frac{\partial u_1}{\partial x_1}+t_{12}\frac{\partial u_1}{\partial x_2}+t_{13}\frac{\partial u_1}{\partial x_3}+t_{12}\frac{\partial u_2}{\partial x_1}+t_{22}\frac{\partial u_2}{\partial x_2}+t_{23}\frac{\partial u_2}{\partial x_3}+t_{13}\frac{\partial u_3}{\partial x_1}+t_{23}\frac{\partial u_3}{\partial x_2}+t_{33}\frac{\partial u_3}{\partial x_3}}$
42. $\overline{\frac{\partial \rho}{\partial x_1}}$
43. $\overline{\frac{\partial \rho}{\partial x_2}}$
44. $\overline{\frac{\partial \rho}{\partial x_3}}$
45. $\overline{\frac{\partial u_1}{\partial x_1}}$
46. $\overline{\frac{\partial u_1}{\partial x_2}}$
47. $\overline{\frac{\partial u_1}{\partial x_3}}$
48. $\overline{\frac{\partial u_2}{\partial x_1}}$
49. $\overline{\frac{\partial u_2}{\partial x_2}}$
50. $\overline{\frac{\partial u_2}{\partial x_3}}$
51. $\overline{\frac{\partial u_3}{\partial x_1}}$
52. $\overline{\frac{\partial u_3}{\partial x_2}}$
53. $\overline{\frac{\partial u_3}{\partial x_3}}$
54. $\overline{\frac{\partial T}{\partial x_1}}$
55. $\overline{\frac{\partial T}{\partial x_2}}$
56. $\overline{\frac{\partial T}{\partial x_3}}$
57. $\overline{u_1\frac{\partial \rho}{\partial x_1}}$

58. $\overline{u_1 \frac{\partial \rho}{\partial x_2}}$
59. $\overline{u_1 \frac{\partial \rho}{\partial x_3}}$
60. $\overline{u_2 \frac{\partial \rho}{\partial x_1}}$
61. $\overline{u_2 \frac{\partial \rho}{\partial x_2}}$
62. $\overline{u_2 \frac{\partial \rho}{\partial x_3}}$
63. $\overline{u_3 \frac{\partial \rho}{\partial x_1}}$
64. $\overline{u_3 \frac{\partial \rho}{\partial x_2}}$
65. $\overline{u_3 \frac{\partial \rho}{\partial x_3}}$
66. $\overline{T \frac{\partial \rho}{\partial x_1}}$
67. $\overline{T \frac{\partial \rho}{\partial x_2}}$
68. $\overline{T \frac{\partial \rho}{\partial x_3}}$
69. $\overline{\rho \frac{\partial u_1}{\partial x_1}}$
70. $\overline{\rho \frac{\partial u_1}{\partial x_2}}$
71. $\overline{\rho \frac{\partial u_1}{\partial x_3}}$
72. $\overline{\rho \frac{\partial u_2}{\partial x_1}}$
73. $\overline{\rho \frac{\partial u_2}{\partial x_2}}$
74. $\overline{\rho \frac{\partial u_2}{\partial x_3}}$
75. $\overline{\rho \frac{\partial u_3}{\partial x_1}}$
76. $\overline{\rho \frac{\partial u_3}{\partial x_2}}$
77. $\overline{\rho \frac{\partial u_3}{\partial x_3}}$
78. $\overline{\rho \frac{\partial T}{\partial x_1}}$
79. $\overline{\rho \frac{\partial T}{\partial x_2}}$
80. $\overline{\rho \frac{\partial T}{\partial x_3}}$
81. $\overline{\kappa \frac{\partial T}{\partial x_1}}$
82. $\overline{\kappa \frac{\partial T}{\partial x_2}}$
83. $\overline{\kappa \frac{\partial T}{\partial x_3}}$

84. $\overline{P\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)}$

85. $\overline{T\left(u_1\frac{\partial \rho}{\partial x_1}+u_2\frac{\partial \rho}{\partial x_2}+u_3\frac{\partial \rho}{\partial x_3}\right)}$

86. $\overline{\rho\left(u_1\frac{\partial T}{\partial x_1}+u_2\frac{\partial T}{\partial x_2}+u_3\frac{\partial T}{\partial x_3}\right)}$

87. $\overline{(u_1u_1+u_2u_2+u_3u_3)\frac{\partial \rho}{\partial x_1}}$

88. $\overline{(u_1u_1+u_2u_2+u_3u_3)\frac{\partial \rho}{\partial x_2}}$

89. $\overline{(u_1u_1+u_2u_2+u_3u_3)\frac{\partial \rho}{\partial x_3}}$

90. $\overline{u_1\left(u_1\frac{\partial \rho}{\partial x_1}+u_2\frac{\partial \rho}{\partial x_2}+u_3\frac{\partial \rho}{\partial x_3}\right)}$

91. $\overline{u_2\left(u_1\frac{\partial \rho}{\partial x_1}+u_2\frac{\partial \rho}{\partial x_2}+u_3\frac{\partial \rho}{\partial x_3}\right)}$

92. $\overline{u_3\left(u_1\frac{\partial \rho}{\partial x_1}+u_2\frac{\partial \rho}{\partial x_2}+u_3\frac{\partial \rho}{\partial x_3}\right)}$

93. $\overline{\rho\left(u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_1}+u_3\frac{\partial u_3}{\partial x_1}\right)}$

94. $\overline{\rho\left(u_1\frac{\partial u_1}{\partial x_2}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_3}{\partial x_2}\right)}$

95. $\overline{\rho\left(u_1\frac{\partial u_1}{\partial x_3}+u_2\frac{\partial u_2}{\partial x_3}+u_3\frac{\partial u_3}{\partial x_3}\right)}$

96. $\overline{\rho\left(u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right)}$

97. $\overline{\rho\left(u_1\frac{\partial u_2}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_2}{\partial x_3}\right)}$

98. $\overline{\rho\left(u_1\frac{\partial u_3}{\partial x_1}+u_2\frac{\partial u_3}{\partial x_2}+u_3\frac{\partial u_3}{\partial x_3}\right)}$

99. $\overline{\rho u_1\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)}$

100. $\overline{\rho u_2\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)}$

101. $\overline{\rho u_3\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)}$

102.

$$\overline{\frac{\partial \rho}{\partial x_1}(u_1u_1+u_2u_2+u_3u_3)u_1+\frac{\partial \rho}{\partial x_2}(u_1u_1+u_2u_2+u_3u_3)u_2+\frac{\partial \rho}{\partial x_3}(u_1u_1+u_2u_2+u_3u_3)u_3+}$$

$$\overline{2\rho\left[\left(\frac{\partial u_1}{\partial x_1}u_1+\frac{\partial u_2}{\partial x_1}u_2+\frac{\partial u_3}{\partial x_1}u_3\right)u_1+\left(\frac{\partial u_1}{\partial x_2}u_1+\frac{\partial u_2}{\partial x_2}u_2+\frac{\partial u_3}{\partial x_2}u_3\right)u_2+\left(\frac{\partial u_1}{\partial x_3}u_1+\frac{\partial u_2}{\partial x_3}u_2+\frac{\partial u_3}{\partial x_3}u_3\right)u_3\right]}$$

$$\overline{\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\rho\left(u_1u_1+u_2u_2+u_3u_3\right)}$$

Text names for the terms are as follows:

1. **stats_time**
2. **x**
3. **y**
4. **z**
5. **r**
6. **u**
7. **v**
8. **w**
9. **T**
10. **P**
11. **mu**
12. **kap**
13. **cp**
14. **cv**
15. **ru**
16. **rv**
17. **rw**
18. **rT**
19. **ruu**
20. **ruv**
21. **ruw**
22. **rvv**
23. **rvw**
24. **rww**
25. **ruT**
26. **rvT**
27. **rwT**
28. **rTT**
29. **ruvmag2**
30. **rvvmag2**
31. **rwvmag2**

32. **t11**
33. **t12**
34. **t13**
35. **t22**
36. **t23**
37. **t33**
38. **t1kuk**
39. **t2kuk**
40. **t3kuk**
41. **tijui_xj**
42. **r_x**
43. **r_y**
44. **r_z**
45. **u_x**
46. **u_y**
47. **u_z**
48. **v_x**
49. **v_y**
50. **v_z**
51. **w_x**
52. **w_y**
53. **w_z**
54. **T_x**
55. **T_y**
56. **T_z**
57. **ur_x**
58. **ur_y**
59. **ur_z**
60. **vr_x**
61. **vr_y**
62. **vr_z**
63. **wr_x**

64. **wr_y**
65. **wr_z**
66. **Tr_x**
67. **Tr_y**
68. **Tr_z**
69. **ru_x**
70. **ru_y**
71. **ru_z**
72. **rv_x**
73. **rv_y**
74. **rv_z**
75. **rw_x**
76. **rw_y**
77. **rw_z**
78. **rT_x**
79. **rT_y**
80. **rT_z**
81. **kapT_x**
82. **kapT_y**
83. **kapT_z**
84. **Puk_xk**
85. **Tukr_xk**
86. **rukT_xk**
87. **vmag2r_x**
88. **vmag2r_y**
89. **vmag2r_z**
90. **uukr_xk**
91. **vukr_xk**
92. **wukr_xk**
93. **rukuk_x**
94. **rukuk_y**
95. **rukuk_z**

96. **ruku_xk**
97. **rukv_xk**
98. **rukw_xk**
99. **ruuk_xk**
100. **rvuk_xk**
101. **rwuk_xk**
102. **turbtrans1**